

# The Optimal Monetary Policy Rule: the Role of Asymmetries and Reputation\*

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## Abstract

I study the optimal policy rule of a central bank under an asymmetric information and reputation building setting. I find that the optimal reaction function is nonlinear, despite a standard quadratic loss function and linear Phillips curve. The asymmetry arises from the signal extraction problem of the agents. Given this setting, I analyze the propositions of Rogoff and Blinder how to solve inflationary bias. I find that in the both cases the optimal policy function is nonlinear. Furthermore, the Blinder setting delivers higher value of social welfare.

## 1 Introduction

As of October 2011, the Bank of England has overshoot its 2% inflation target for 22 months in a row. At the same time, in the letter to the Chancellor of the Exchequer, the governor of the bank Mervyn King argued that the bleak economic outlook over the medium run required to take further expansionary measures.

There are two possible explanations for this track record. Either the forecasts of the bank turned out to be overoptimistic, or the central bank has not been following its symmetric target. While the length of the period would suggest the latter, due to the huge uncertainty about the global macroeconomic outlook, the former explanation cannot be ruled out. In practice, the public cannot be sure which alternative is causing the difference between the central bank's announced target and realized inflation.<sup>1</sup>

In this paper, I explicitly model this uncertainty. In particular, the central bank announces the policy it plans to follow and the representative agent observes only a noisy outcome of the actions of the central bank and therefore she cannot directly observe whether the bank actually follows its announcement or not.

The main conclusion of my paper is that the optimal policy rule can be nonlinear even if the loss function of the central bank is quadratic. In other words, estimating a nonlinear policy rule is not sufficient to conclude that the central bank has asymmetric preferences.

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<sup>1</sup>In August 2011, the Inflation Attitudes Survey conducted by the Bank of England showed that 46% of people think that the inflation was "about right", while the median expected inflation in 12 months was 3.5%, so well above the target.

By demonstrating this possibility, I contribute to the recent literature on asymmetric preferences of the central bank and inflationary bias. Nobay and Peel (2003), Cukierman and Gerlach (2003) and Cukierman and Muscatelli (2008) showed that an asymmetric loss function gives rise to inflationary bias even if the central bank's targets the natural level of output. The standard approach in this literature is to estimate the loss function from the policy choices of the central bank and then argue about the size of the inflationary bias. While this literature does not explicitly propose a policy to reduce this bias, it is clear that any such policy would be welfare detrimental if the asymmetric behavior was not driven by asymmetric preferences. Here, I argue that observed nonlinearities and existence of inflation bias can be addressed without altering the standard preferences. While there is no reason why a symmetric quadratic loss function is necessarily the right description of central bank's preferences, it is important to understand if and under what circumstances asymmetric policy functions can arise even without assuming an asymmetric loss function.

In my model, the central bank minimizes the expected discounted sum of a standard quadratic loss function, which has its bliss point above the natural level of output. This social welfare function is known and shared with the representative agent. The central bank announces a policy it plans to implement and the representative agent adopts this policy as a basis of her inflation expectations. Given the value of supply shock, the central bank then chooses an allocation, not necessarily following the announcement. However, the representative agent observes only a noisy signal about this allocation and hence she has to use signal extraction to infer whether the central bank has followed its announcement or whether it has deviated. If the representative agent suspects the central bank of deviating, she revises the inflation expectations to the value given by the known parameters of the social welfare function.

The optimal behavior of the central bank turns out to be nonlinear due to the fact that private agents have imperfect information about the outcomes chosen by the central bank. The central bank exploits this informational advantage, but it is constrained by the fact that the optimal policy still has to deliver in expectation the same outcome as would the announced policy. The central bank is thus more accommodative in situations when such a deviation is most profitable while it is more hawkish at other times so that on average the deviations cancel out.

The paper proceeds as follows. Section 2 summarizes the relevant literature. Section 3 formulates the model. Section 4 explains the practical implementation of the model. Section 5 discusses the choice of the parameters. Section 6 shows the results. Section 7 discusses the robustness of the results and finally, section 8 concludes.

## 2 Overview of the literature

In this section, I provide a broad overview of related literature. I divide the contribution to three areas; the inflation bias in the traditional setting, models of reputation with asymmetric information and new inflation bias theory.

The first area is important because all the features of the traditional inflation bias model based on the seminal contribution of Kydland and Prescott (1977) appear in my model. In this section, the most relevant piece of research from this strand of literature is Lohmann (1992), who also showed how a nonlinear policy rule can arise with linear Phillips curve. The difference between my paper and hers is in the mechanism of how the nonlinearity arises. In my paper, it is due to the imperfectly informed private agents, whereas in Lohmann (1992) this result is driven by the tension between the central banker, who is more conservative than the rest of the society and a discretionary politician with power to remove the central banker from the office at some fixed cost.

The second area is relevant because it provides insights into the reputation building mechanism. The main difference to my paper is that the focus in this literature lies on modelling situations, where the central bank observes only a noisy signal of economic fundamentals before choosing the allocation. By doing so, this literature focused more on the effects of uncertainty and lags in the transmission mechanism for credibility. In my model it is the public who cannot perfectly observe the action of the central bank. Secondly, this line of research is the base for the literature on the role of transparency for the conduct of monetary policy. As discussed in detail below, in my model, an increase in transparency would lead to a reduction in the social welfare.

The third area of literature relates to the asymmetric loss function. The fact that asymmetric loss functions would generate nonlinear policy function together with some empirical evidence on the latter is sometimes presented as an argument that the central bank's loss function is in fact asymmetric. In this sense my paper provides an alternative explanation of how the nonlinear policy rules arise without assuming an asymmetric loss function.

### 2.1 Traditional inflationary bias

My model builds on the simplest possible setting with inflationary bias. This literature was initiated by Kydland and Prescott (1977), who first noticed that even a perfectly benevolent policy maker with the same preferences as the rest of the society might not deliver an optimal outcome. The reason for this failure is that the optimal monetary policy plan is not sub-game perfect. In this setting, the central bank takes the inflation expectations as given when it chooses the inflation-output gap allocation. After the inflation expectations have been fixed, the central bank might find it optimal to increase the inflation above the expected level in order to boost output and thus deviate from the *ex ante* optimal policy. Perfectly rational agents foresee this lack of commitment and set their inflation expectations to a value where it is not profitable for the central bank to push inflation even higher. The outcome is positive inflation while the output is at its natural level. This situation can be avoided if the central bank can credibly commit itself to following some rule. Without commitment, the inflationary bias arises.

The insights of Kydland and Prescott led to voluminous subsequent research.<sup>2</sup> Barro and Gordon (1983) showed that the inflationary bias can be eliminated even without any commitment

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<sup>2</sup>For general overview and discussion of the inflationary bias literature see Walsh (2001, chapter 8).

if the model is viewed as an infinite repetitive game. The “cooperation” is enforced by a reputation building repeated game.

Rogoff (1985) suggested that the bias can be reduced if the central bank is more hawkish than the rest of the society, or in another words, if the central bank is more penalized for inflation deviations from its target than the general public. There are important reservations to the solution proposed by Rogoff. First, within the model, the change in the preference parameter generates a wedge between the policy which would be chosen normally and the one which would be chosen by the conservative central banker. The bigger is the supply shock (in absolute terms), the bigger is the difference in allocations and hence the bigger is the cost paid for lowering the inflationary bias. In another words, the conservative central banker implements a suboptimal trade-off between inflation and output stabilization. Second, Rogoff did not consider the credibility of the arrangement, which might be an issue if the central bank is not perfectly independent. Finally, from a methodological perspective, introducing a conservative central banker is a technical short-cut in the sense that given the preferences of the society, the optimal monetary policy should be modelled as the optimal solution to the central bank problem.

Flood and Isard (1988) observed that the costs of installing a Rogoff style conservative central banker are increasing in the magnitude of the supply shock. As a solution they suggested an exit clause which would allow a policy maker (a politician appointing the central banker) to temporarily remove the conservative central banker from power at some fixed cost for the policy maker. For low magnitude shocks, the conservative central banker successfully implements hawkish monetary policy, extreme shocks are accommodated by the policy maker and the credibility of the central bank is preserved.

More recently, Lohmann (1992) extends the model of Flood and Isard by modeling the interplay between the policy maker and the central banker. The central banker would prefer setting a hawkish allocation, but if such policy is chosen, it would be overridden by the policy maker. However, as the central banker knows the (fixed) cost of removing her from office, she can set her policy such that marginal benefits of invoking the exit clause are zero. Such policy is still more preferable for the central banker than the allocation that would be chosen by the policy maker alone. In equilibrium, the central banker is never removed from the office. Lohmann shows that this setting generates a nonlinear policy rule. Lohmann’s contribution is significant to the problem outlined in this paper, because it shows how an asymmetric reaction function can arise even in a setting with quadratic loss function and a linear Phillips curve. However, the solution proposed by Lohmann still relies on the short-cut notion of the conservative central banker, whereas my model is fully micro-founded. Furthermore, Lohmann’s solution neither completely removes the inflationary bias nor is it true that the trade-off between inflation and output is optimal. In my setting, however, both of these issues are addressed by allowing for any possible trade-off and any possible output target announcement. Lastly, the assumption that politician can sack the central banker is becoming harder to defend as central banks have become more independent over the last two decades.

Walsh (1995) suggested that a suitably written contract with the governor of the central bank might credibly prevent the central bank from discretionary behavior. The optimal contract can be difficult to implement from both theoretical (it might depend on potentially unobservable characteristic of the central banker) and practical point of view (the performance bonus is likely

to be low compared to any outside option the governor might have).

An alternative way to prevent inflationary bias is if the central bank targets the natural rate of output. In that case the inflationary bias does not arise because the ex ante policy is no longer sub-game imperfect. This approach is sometimes called *target* conservative (as opposed to the term *weight* conservative that refers to changing the weights in the loss function proposed by Rogoff (1985)). The implication of this stream of literature was to increase the independence of the central bank and hence insulate the monetary policy from political influence. Blinder (1998) argues that because the central bankers know the literature on inflationary bias, they do not attempt to target the socially optimal level of output and hence the discretionary policy does not induce the inflationary bias, because there is no incentive to deviate and boost the economy. While it is true that the central bank now implements the correct trade-off between inflation and output stabilization, it is still the case that Blinder's suggestion is merely a shortcut delivering a reasonable behavior rather than a truly micro-founded solution. As such, it in some sense resembles calvo pricing; it provides a computationally simple setting which captures the main behavioral aspects at the cost of not modeling the microfoundations. The fact that the observed behavior is close to the one which would be implemented by Blinder's central bankers does not necessarily constitute that the central banker in charge has preferences with the bliss point at the natural level of unemployment/output.

There have been attempts to estimate the existence and importance of the inflationary bias empirically. Ireland (1999) uses a version of the Kydland-Prescott model and tests its predictions on inflation and unemployment data of the US since 1960's. He assumes that the natural rate of unemployment follows a unit root process. The result of the central bank optimization implies that inflation is cointegrated with the natural rate of unemployment. Ireland then tests the restrictions generated by the cointegration relation and finds that the long run predictions of the inflationary bias model are supported by the data. Indeed, when looking at smoothed inflation time series in the US, there is a clear difference between the early post-war period of 1960's to early 1980's, where the smoothed time series were trending up, and from late 1980's onwards, where the trend was reversed and inflation rates started to decline.

However, the predictions of the model for the short run are rejected. Ireland argues that this is not surprising and that this result should not be used as evidence against the models as such: "*Given the large number of restrictions... it comes as no surprise that statistical tests reject the constrained model. After all, the underlying theory is described by three simple equations... it would be unreasonable to expect this simple structure to account for all of the dynamics that can be found in the data.*" (Ireland, 1999, page 289). The original models of Kydland and Prescott and Barro and Gordon were not meant to explain short term fluctuations in inflation. Furthermore, the way they capture the gap between the target and the natural rate is constructed to be a simple representation. The fact that the inflationary bias depends on the absolute value of the unemployment which is modeled as time varying is thus rather an artifact of the adapted mathematical formalism, rather than a deliberate intention of the original authors.

## 2.2 Models of reputation with asymmetric information

The second main ingredient of my model is the informational asymmetry between the representative agent and the central bank. I model the central bank as observing the true shocks and the public as observing only a noisy signal of the allocation chosen by the central bank. Romer and Romer (2000) support this view by documenting that the forecasts of the FED were much better than the forecast of all other entities. I abstract from intertemporal aspects introduced by imperfect control. However, in the spirit of Romer and Romer's results, the central bank is still better informed about the economic fundamentals than the representative agent.

The literature so far has typically focused on settings where the central bank does not observe the economic fundamentals perfectly. Such models hence try to capture a different aspect of reality than my model. However, many insights about the mechanism of reputation building for repeated monetary policy games with asymmetric information are present in my model as well so it is worth summarizing and contrasting the main contributions. Furthermore, reviewing the literature on imperfect information sheds light on why asymmetry arises in my setting while it does not typically arise in the other models with asymmetric information.

Canzoneri (1985) commented on Barro and Gordon's results by observing that the reputation building is complicated in the presence of imperfect information. In his model, the goal of the central bank is to accommodate shocks in money demand by printing money. However, the central bank observes only a noisy signal about future money demand. The private agents observe the final allocation and decide whether they believe that the outcome is due to a deliberate attempt to inflate the economy or only a result of a mistaken forecast by the central bank.

Canzoneri constructs the equilibrium in a similar fashion as in Barro and Gordon (1983). However, he shows that in order to enforce the cooperative equilibrium, the agents will set up a threshold level of inflation such that they only interpret excessive inflation as an attempt of cheating if it exceeds this threshold. In other words, the threshold governs how bad a forecast the agents are willing to tolerate before starting to suspect that the central bank has deviated. Interestingly, in equilibrium it is never optimal for the central bank to try to cheat. Yet because the forecast error is stochastic, there are some periods where agents trigger the punishment.

Given the context of nonlinear loss and policy functions, it is worth noting that there cannot be any asymmetry in the reaction function in Canzoneri's setting simply because the bank sets its policy before the uncertainty is realized. Although closely related, my paper differs from that of Canzoneri in one crucial aspect. Canzoneri assumes that the central bank has imperfect information about future demand. In my model, the central bank has complete knowledge of fundamentals and it is the agents who observe only a noisy signal about the allocation chosen by the central bank (the noise shock is realized only after the central bank has chosen the allocation). The choice of the uncertainty assumption differs because the research question differs. Canzoneri studies the role of imperfections in the implementation of the monetary policy on the reputation building whereas this paper explores the setting where the central bank is better informed than the members of the public.

Backus and Driffill (1985) explore the implications of the agents' uncertainty about the preferences of the central bank. In their model, there are two types of central banks which differ in terms of preferences towards inflation. The agents update their beliefs based on the observed outcomes.

Backus and Driffill show that even a central bank of an inflationist type might find it profitable to pretend that its type is hawkish. The model also predicts that the disinflation policies would be more costly for a bank with bad reputation. Backus and Driffill's are close to my model, because in their model the agents do not observe the true nature of the central bankers, in my model the economic agents do not observe the true behavior of the central bank. However, my setting is more general, because I allow the central banker to pretend any degree of conservativeness, whereas Backus and Driffill restrict the number of types to two.

Cukierman and Meltzer (1986) effectively combine both information asymmetries of Canzoneri's and Backus-Driffill's models. They explore a setting where the agents observe neither the direct action of the central bank nor the time-varying preferences of the central bank. The public observes the outcomes, updates its beliefs about the state of the central bank's preferences and adjusts its inflation preferences accordingly. Under this setting, Cukierman and Meltzer show that using the most precise instrument is not necessarily the instrument minimizing the loss of the central bank. There are two ingredients to this result. First, the more precise the instrument is, the faster the public learns about the preferences of the central bank by observing its actions. Second, the stochastic and persistent preferences of the central bank make the relative benefits of inflating and boosting the economy time varying. With these two combined, when the central bank is hit by a preference shock which increases its desire to inflate the economy, a relatively less precise instrument means that the public will need more time to learn about the shift in the central bank's preferences and will ultimately adjust the inflation expectations less and later. A less precise instrument thus slows down the reaction of the inflation expectations making the expansionary policy more efficient.

Cukierman and Meltzer's paper, the central bank exploits the fact that the agents do not know the true preferences of the central bank. While this is also true in my paper, the crucial difference is that in their paper, the central bank preferences are stochastic, whereas in my paper, they are fixed and known to the public. The stochastic nature of the central bank preferences can be interpreted as a shortcut to capture other changes in the economy, as for example financial fragility, which are not explicitly modelled. Otherwise, there is nothing explicit in their model which would suggest link the changes the preferences of the central bank to the welfare of the representative agent. This is different in my model, as the central bank exploits its informational advantage so it reacts differently to different shocks which directly affects the welfare of the representative agent.

Cukierman and Meltzer can be viewed as one of the first papers studying the role of transparency in monetary policy. This stream of literature became very influential as more central banks adopted inflation targeting as a framework for conducting monetary policy. For an overview of the literature on transparency, see Geraats (2002). In a review chapter, Stokey (2003) takes Canzoneri's insight and analyzes the optimal instrument choice. If there are two instruments available, it is generally better to use the one which has higher correlation with the target variable. However, if the other one is much more observable and hence the private agents can use it in order to verify central banks actions, Stokey argues that it might be better to use the less correlated, but more transparent instrument.

My model relates to the literature on transparency of monetary policy in an indirect way. My assumptions allow the central bank to partially exploit the information advantage over the representative agent. By doing so, the central bank delivers higher social welfare. In my model,

the increase in transparency would decrease the information advantage of the central bank and eliminate the possible gains from deviations which are not detected by the representative agent.

### 2.3 Asymmetric loss function and new theory of inflationary bias

The new literature on inflationary bias shows that this bias can arise in a setting where the central bank is targeting the natural rates, but the loss function is asymmetric.

Nobay and Peel (2003) explore the implication of having an asymmetric loss function and study the situation where the loss function has a linex form.<sup>3</sup> They note that with asymmetric preferences, it is not necessarily the case that targeting the natural rate is welfare maximizing, depending on the skewness of the loss function and the variance of the supply shock.

Cukierman and Gerlach (2003) and Cukierman and Muscatelli (2008) provide a deeper economic intuition to the argument of Nobay and Peel. Their explanation of the bias is as follows: Consider a central bank which targets the natural rate of unemployment, but has an asymmetric loss function, so unemployment above the target is relatively more harmful than a deviation of the same magnitude in the opposite direction. Second, assume that the central bank chooses the policy before observing the supply shock. Given the skewed expected losses, the central bank prefers to err on the negative side of unemployment gap and hence chooses an accommodative policy. The public recognizes this point and incorporate this effect into inflation expectations, making the inflationary bias even bigger. The model has testable implications, because it predicts that the bigger the variance of supply shocks is, the bigger is “precautionary” action taken by the central bank and hence the bigger is the resulting inflationary bias.

Cukierman and Muscatelli (2008) estimate a nonlinear policy rule using smooth transition regression with a hyperbolic tangent function, documenting a concave Taylor rule for the UK prior to adoption of inflation targeting. Cukierman and Muscatelli also report an interesting switch in the asymmetry with respect to inflation in the US, where the McChesney-Martin tenure at the FED could be characterized as having asymmetrically higher losses for high inflation, whereas under Greenspan’s chairmanship the FED could be characterized as having asymmetrically higher losses from output gap.

? tests whether the inflationary bias is better explained by the standard setting where central banks target positive output gap, or rather where the inflationary bias is caused by the asymmetric preferences of the central bank. The linex loss function nests the quadratic preferences as a special case, so it is possible to discriminate the two based on testing restrictions on the parameters in the functional form. The identification relies on the fact that past unemployment is informative for future inflation; this relation is linear for the standard inflationary bias model, whereas it is nonlinear for the Cukierman model. Ruge-Murcia finds that the standard inflationary bias is rejected by the data, whereas the Cukierman model is not. ? extends the analysis to Canada, France, Italy, Japan, UK and the US and obtains mixed results. However, as pointed out by Ireland (1999), the Barro-Gordon model is a very simple model and hence it would be too ambitious to require it to explain fully short term swings in inflation.

Surico (2007) explores the asymmetric preferences of the FED. He starts with a general loss function where the quadratic terms in unemployment and inflation are exchanged by linex func-

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<sup>3</sup>Linex form:  $g(x) = \frac{\exp(\gamma x) - \gamma x - 1}{\gamma^2}$ , where  $x = u - ku^*$ ,  $u^*$  is the natural rate of unemployment.



tions. He then tests whether and which part of the loss function is asymmetric and during which time period. He finds the FED's loss function to be asymmetric with respect to output from 60's to early 80's, but there is no asymmetry detected with respect to inflation in the whole sample. Surico thus finds that the Cukierman's style inflationary bias is present only in the pre-Volcker period and that it caused an increase of inflation by 1.48 percentage points. Surico (2008) confirms these results using a slightly different methodology, finding that the implicit inflation target declined from 3.81% to about 2%, the inflationary bias was about 1 percentage point in the period from 1960's to early 1980's and the bias disappeared in the period starting in the late 1980's.

Dolado et al. (2004) analyze the optimal monetary policy rule for a setting where the central bank has a linear loss function and the Phillips curve is also nonlinear. The estimates of the model on US data suggest that the Phillips curve is linear, but there is nonlinearity in the loss function of the FED after 1983. Dolado et al. (2005) extend the analysis to data from France, Germany, Spain, the Eurozone as a whole and to the US. The main finding is that the central banks behave nonlinearly since 1980's, reacting more strongly to deviations above the targets than to deviations below the targets in inflation and output.

These contradicting empirical results (Surico and Dolado et al.) suggest that there is no clear consensus regarding the asymmetry in the central banks' loss function. Generally, there is a tendency to estimate the policy functions separately for periods of different chairmen of the FED which conflicts with GMM's bad small sample properties. Furthermore, even with unlimited amounts of data, the estimation of forward looking policy rules suffers from the weak instrument problem and the identification issues (Mavroeidis, 2010). Another issue is the fact that there is no limit on the possible functional form that the nonlinear policy rule can take and this introduces another degree of freedom in the estimation. This concern is addressed by Osborn et al. (2005), who apply Hamilton (2001)'s framework which allows to relax strict parametric assumptions about the shape of the estimated nonlinear function. They find that the interaction between inflation and output gap should be included in the specification of the policy rule.

Overall, it is fair to say that the existing empirical results do not provide a fully convincing evidence for or against asymmetric loss functions. Therefore there is scope in exploring other possible sources of asymmetry in the policy functions. This paper offers such an alternative.

### 3 Model

In this section I describe my model. First, I formally describe the differences between the textbook setting and my model (section 3.1). Then I will describe the textbook results describing the optimal policy of a discretionary central bank (section 3.2). Then, given these results, I solve the signal extraction problem of the representative agent (section 3.3). Given the results of the signal extraction exercise, I describe the problem of the central bank and show how to solve it using value function iteration (section 3.4). Finally, I discuss the assumptions regarding the representative agent's rationality and learning (section 3.5).

### 3.1 Basic setting

There are two players in the economy: a central bank and a representative agent. The economic environment is characterized by the inflation expectations augmented Phillips curve,

$$y_t = \alpha(\pi_t - \pi_{t|t-1}) + \varepsilon_t, \quad (1)$$

where  $y_t$  and  $\pi_t$  denote the output gap and the inflation in period  $t$ ,  $\pi_{t|t-1}$  represents the expectations formed by the representative agent in period  $t - 1$  about the inflation in period  $t$ . Both the central bank and the representative agent have the same loss function:

$$L_t = (y_t - y_h^*)^2 + \beta_l \pi_t^2, \quad (2)$$

where  $y_h^*$  is the socially optimal level of output. This specification implies that the output target is at the socially optimal level of output  $y_h^*$ . This value is higher than the natural level of output (which is normalized to zero).  $y_h^*$  might be positive for example due to the monopolistic competition on the markets leading to higher prices and lower output than what a social planner would choose. The preference parameter  $\beta_l$  is also common for both the central bank and the rest of the society.

The subscripts of  $\beta$  and  $y^*$  already indicate the direction of the solution to the inflationary bias problem. A conservative central banker would be described by a preference parameter which puts *higher* emphasis on inflation stabilization ( $\beta_h > \beta_l$ ). Blinder (1998) conjectures that central banks target *lower* output than the socially optimal value,  $y_l^* < y_h^*$ . Using subscripts for the true preferences parameters as well as for the announcement values allows to make calculations for generic values of  $(\beta, y^*)$  more transparent.

Here, however, the central bank cannot decide to change its preferences parameters  $y_h^*$  and  $\beta_l$ , as conjectured by Rogoff or Blinder. Instead, the central bank *announces* that it plans to behave according to some  $\beta_h$  and  $y_l^*$ . The representative agent understands that the true preferences are described by  $(\beta_l, y_h^*)$  and hence she tests every period if the central bank behaves according to its announcement rather than the true preference parameters.

The central bank minimizes the discounted expected loss (assuming the economy starts at  $t = 0$ ) using a discount factor  $\delta$ :

$$L = \mathbb{E} \sum_{t=0}^{\infty} \delta^t L_t$$

The representative agent observes neither the supply shock faced by the central bank nor the chosen allocation (as one would be perfectly informative about the other). She observes only a noisy signal about the allocation chosen by the central bank. The reputation building mechanism then works in the following way: the central bank announces a policy  $(\beta_h, y_l^*)$  and the representative agent trusts the central bank, as long as she does not observe an outcome which would suggest that the central bank has deviated (from behavior which would be optimal given  $(\beta_h, y_l^*)$ ). The details on the deviation detection are given in section 3.3.

The exact timing is the following:

1. The supply shock  $\varepsilon_t$  is realized and observed by the central bank.
2. The central bank chooses an allocation  $y_t = y(\varepsilon_t), \pi_t = \pi(\varepsilon_t)$ .
3. The noise shocks  $\xi_{\pi,t}, \xi_{y,t}$  are realized and the representative agent observes

$$\tilde{y}_t = y_t + \xi_{y,t}, \tag{3}$$

$$\tilde{\pi}_t = \pi_t + \xi_{\pi,t}. \tag{4}$$

4. The representative agent faces a signal extraction problem and infers the likelihood of the central bank *deviating*, denoted by  $l_D(\tilde{y}_t, \tilde{\pi}_t)$ , or *not deviating*, denoted by  $l_{ND}(\tilde{y}_t, \tilde{\pi}_t)$ , and revises the inflation expectations  $\pi_{t+1|t}$  for the next period accordingly:

- if she believes that the central bank has not deviated, she sets her expectations of future inflation as  $\pi_{t+1|t} = \pi(\beta_h, y_l^*)$ ,
- if she believes that the central bank has deviated, she sets her expectations of future inflation as  $\pi_{t+1|t} = \pi(\beta_l, y_h^*)$ .

Choosing  $\pi_{t+1|t} = \pi(\beta_l, y_h^*)$  will be called *triggering punishment*, because it effectively punishes the central bank for deviating in the current period by increasing the loss in the next period (higher inflation expectations worsen the output gap - inflation trade-off). The punishment is either forever, or only for a finite number of periods. The finite punishment can be interpreted as if the governor loses his job after the public loses the confidence in the central bank.

For the sake of exposition, I will consider that there is only one possible deviation, i.e. the central bank can behave either according to  $(\beta_h, y_l^*)$  or deviate by behaving according to  $(\beta_l, y_h^*)$ .<sup>4</sup> The representative agent always tests the hypothesis of no deviation versus the hypothesis of complete deviation.

The case with only two possible actions by the central bank and eternal punishment is depicted in figure 1. Figure 1 uses the following notation:  $L_{ND}(\varepsilon_t)$  denotes the value of the loss function when the central bank does not deviate while facing the supply shock  $\varepsilon_t$ . Similarly,  $L_D$  is the value of the loss function if the central bank deviates.  $V^P$  and  $V^{NP}$  are the value functions of being either in the *punished* or in *not punished* state, respectively.

## 3.2 Basic Algebra of the model

It is convenient to derive the basic results of the textbook setting of the inflationary bias. Doing so helps to develop the intuition and to build the tools which are used later. In this section I present only the basic results, the details on derivation can be found in appendix B.

### 3.2.1 Basic setting

Let's start with a generic loss function:

$$L_t = \frac{1}{2} (y_t - y^*)^2 + \frac{\beta}{2} \pi_t^2, \tag{5}$$

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<sup>4</sup>This assumption is released later. The numerical results presented are computed allowing the central bank to deviate on a very fine grid.

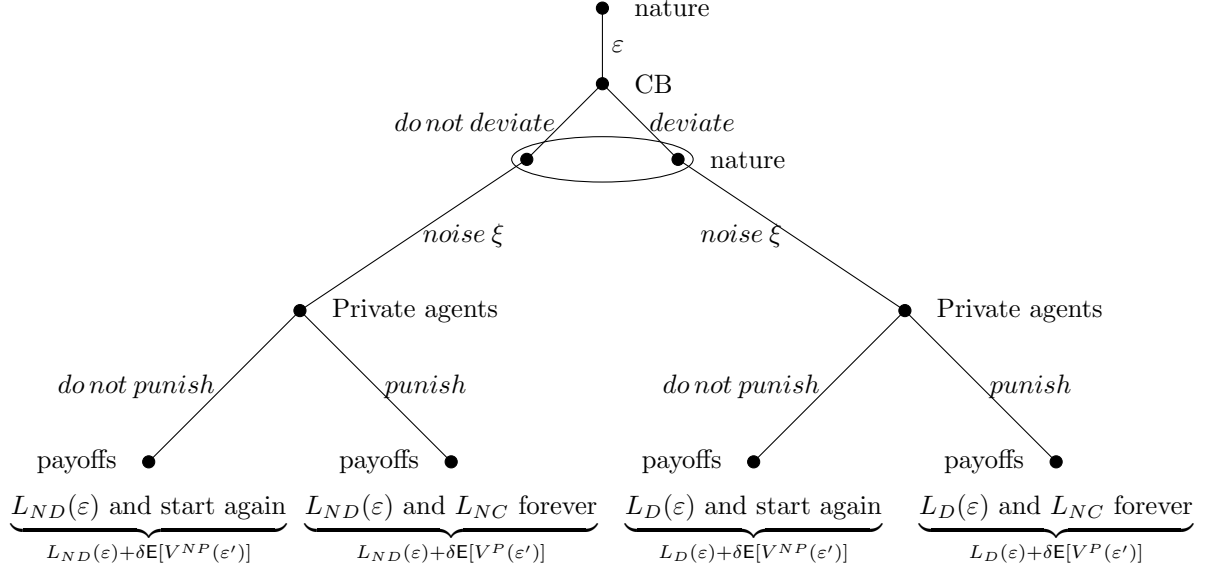


Figure 1: Game scheme, starting from the not punished stage. The central bank deviates either fully or not at all. Once the representative agent revises her beliefs, the beliefs stay revised forever.

where  $y$  is output gap,  $y^* \geq 0$  allows for a positive output gap target,  $\pi_t$  is inflation (inflation target normalized to 0) and  $\beta$  is the coefficient of inflation aversion. The inflation expectations augmented Phillips curve is:

$$y_t = \alpha(\pi_t - \pi_{t|t-1}) + \varepsilon_t,$$

where  $\pi_{t|t-1}$  is the expected inflation and  $\varepsilon_t$  denotes the supply shock. I assume that the central bank can choose any combination of output gap and inflation consistent with the Phillips curve and abstract from modelling the policy instrument.

The regime in which the central bank cannot credibly pre-commit to follow some policy rule is called discretion. The inflation expectations are thus not a function of the policy/preferences of the central bank, but they are preset and the central bank acts only after the inflation expectations are realized. For any given inflation expectations, using the first order conditions of the central bank problem, the optimal reaction to the supply shock  $\varepsilon_t$  can be found to be (the superscript  $d$  stands for discretion):

$$\pi_t = \frac{\alpha}{\alpha^2 + \beta} y^* + \frac{\alpha^2}{\alpha^2 + \beta} \pi_{t|t-1} - \frac{\alpha}{\alpha^2 + \beta} \varepsilon_t, \quad (6)$$

$$y_t = \frac{\alpha^2}{\alpha^2 + \beta} y^* - \frac{\alpha\beta}{\alpha^2 + \beta} \pi_{t|t-1} + \frac{\beta}{\alpha^2 + \beta} \varepsilon_t. \quad (7)$$

Under rational expectations, the expectation of inflation must be consistent with the true data

generation process:

$$\pi_{t|t-1} = \mathbb{E}[\pi_t] \quad (8)$$

then

$$\pi_{t|t-1} = \frac{\alpha}{\beta} y^*. \quad (9)$$

The parameters  $(\beta, y^*)$  will reflect the beliefs of the representative agent and I will write the inflation expectations as a function of the beliefs,  $\pi_{t|t-1} = \pi_{t|t-1}(\beta, y^*)$ , in order to emphasize the role of beliefs whenever it is appropriate.

In order to obtain the optimal choice of inflation given any value of the supply shock  $\varepsilon$ , let's insert the equation (9) into (6) and use the Phillips curve (1) to obtain (the superscript *ed* stands for equilibrium under discretion):

$$\pi_t^{ree} = \frac{\alpha}{\beta} y^* - \frac{\alpha}{\alpha^2 + \beta} \varepsilon_t \quad (10)$$

$$y_t^{ree} = \frac{\beta}{\alpha^2 + \beta} \varepsilon_t. \quad (11)$$

and we can notice that the expectation of inflation in equation (10) is indeed the inflation expectation  $\pi_{t|t-1}$ . From here we can observe that  $\frac{\partial \pi_t^D}{\partial \varepsilon_t} < 0$ , so a very inflation averse central bank ( $\beta \rightarrow \infty$  in the loss function) does keep inflation at zero all the time. In such a regime, the inflation would have zero variance and all shocks would be accommodated by movements in the output gap. On the other hand, a central bank that does not care about inflation stabilization at all would be modelled as having  $\beta \rightarrow 0$ .

The superscript *ree* captures the fact that these are the optimal allocations in the rational expectation equilibrium, by which I mean that the inflation expectations of the representative agent are correct in the sense that the equation (8) is satisfied.

### 3.2.2 Expansion path parametrization

The set of the optimal choices of the central bank is called *expansion path* and it forms a straight line. This fact is useful for understanding the mechanics of the signal extraction problem of the representative agent. To see that this is the case, start with equation (6) and the Phillips curve (1) to cancel out the inflation expectations. The expansion path is then

$$y_t = y^* - \frac{\beta}{\alpha} \pi_t. \quad (12)$$

Note, that this expansion path does not depend on  $\pi_{t|t-1}$ , so it holds irrespective of whether the inflations expectations capture correctly the preference parameters  $(\beta_h, y_l^*)$  in the central bank loss function or not.

### 3.2.3 Introducing conservative central banker

For generic preferences  $(\beta, y^*)$ , the equilibrium is described by equations (9), (10) and (11), where the generic values are replaced by the correct values of the preference parameters, i.e. for the not deviating central bank by  $(\beta_h, y_l^*)$  and  $(\beta_l, y_h^*)$  for the central bank which is being punished.

Imagine for the moment that any deviation is immediately detected. Still, the inflation expectations are fixed for the current period and the punishment starts only from the next period onwards. Hence, there are three possible situations:

1. the central bank is not deviating
2. the central bank is deviating for the first time
3. the central bank has deviated in the past and it is now being punished

In order to correctly set up the problem of the central bank of whether or not to deviate and if so by how much, it is important to understand the ramifications of the decision to deviate. In particular, the knowledge of expected value of the loss function in the *not punished* and in the *punished* state are important in order to decide whether the immediate gain from deviation is worth the risk of moving to the punished state. Due to the noise, the chance of detection is not perfect and at the same time, even if the central bank is not deviating, there is still a chance that the agents might punish it, if the noise realization is particularly bad.

**Not Deviating equilibrium** Assume that the central bank with preferences  $(\beta_l, y_h^*)$  acts as if it had preferences  $(\beta_h, y_l^*)$ . In equilibrium, the representative agent sets her inflation expectations according to the behavior  $(\beta_h, y_l^*)$  (and not the actual preferences  $(\beta_l, y_h^*)$ ), so the central banker implements an equilibrium allocation (as solved in equations (11) and (10)):

$$\pi_t^{ND}(\beta_h, y_l^*) = \frac{\alpha}{\beta_h} y_l^* - \frac{\alpha}{\alpha^2 + \beta_h} \varepsilon_t, \quad (13)$$

$$y_t^{ND}(\beta_h, y_l^*) = \frac{\beta_h}{\alpha^2 + \beta_h} \varepsilon_t. \quad (14)$$

Let's denote the value of the loss by  $L_{ND}$ :

$$\begin{aligned} L_{ND}(\varepsilon_t) &\equiv \frac{1}{2} \left( y_t^{ND}(\beta_h, y_l^*) - y_h^* \right)^2 + \frac{\beta_l}{2} \left( \pi_t^{ND}(\beta_h, y_l^*) \right)^2 \\ &= \frac{1}{2} \varepsilon_t^2 \left( \left( \frac{\beta_h}{\alpha^2 + \beta_h} \right)^2 + \beta_l \left( \frac{\alpha}{\alpha^2 + \beta_h} \right)^2 \right) - \varepsilon_t \left( \frac{\beta_h}{\alpha^2 + \beta_h} y_h^* + \beta_l \frac{\alpha^2}{\beta_h(\alpha^2 + \beta_h)} y_l^* \right) \\ &\quad + \frac{1}{2} \left( (y_h^*)^2 + \beta_l \left( \frac{\alpha}{\beta_h} \right)^2 (y_l^*)^2 \right) \end{aligned}$$

and the expected value of not deviating is thus

$$\mathbb{E}L_{ND} \equiv \mathbb{E}[L_{ND}(\varepsilon_t)] = \frac{1}{2} \sigma_\varepsilon^2 \frac{\alpha^2 \beta_l + \beta_h^2}{(\alpha^2 + \beta_h)^2} + \frac{1}{2} \frac{\alpha^2 \beta_l (y_l^*)^2 + \beta_h^2 (y_h^*)^2}{\beta_h^2}. \quad (15)$$

The first term shows that the loss depends on the variance of the supply shock due to the convexity of the quadratic loss function.

To see the effects of different announcements, let's take partial derivatives,

$$\frac{\partial \mathbf{E}[L_{ND}(\varepsilon_t)]}{\partial \beta_h} = \sigma_\varepsilon^2 \frac{\alpha^2(\beta_h - \beta_l)}{(\alpha^2 + \beta_h)^3} + \frac{\alpha^2 \beta_l (y_l^*)^2}{\beta_h^3} \quad (16)$$

$$\frac{\partial \mathbf{E}[L_{ND}(\varepsilon_t)]}{\partial y_l^*} = \frac{\alpha^2 \beta_l}{\beta_h^2} y_l^* \quad (17)$$

The first term in the first equation shows that having  $\beta_h > \beta_l$  creates a wedge between the optimal allocation given the true preferences and the announcement. The wedge is bigger the bigger the shock is, so higher  $\sigma_\varepsilon^2$  exacerbates this problem. The second term in the first equation shows the effect of  $\beta_h$  on the inflationary bias; the bias is present only if  $y_l^* > 0$  and it is reduced by having higher values of the parameter  $\beta_h$  (similar to having a more conservative central banker). The same applies for the second equation.

**Equilibrium in the punished state** In this state, the central bank has no longer any incentive to deviate and the representative agent has correct expectations set according to the true preference parameters  $(\beta_l, y_h^*)$ . The superscript  $P$  captures the fact that this is the equilibrium in the punished state. In this stage, the central bank cannot affect the representative agent's beliefs anymore and hence chooses the discretionary allocation given its true preferences  $(\beta_l, y_h^*)$ . It follows from (11) and (10) that the optimal allocation is

$$\pi_t^P(\beta_l, y_h^*) = \frac{\alpha}{\beta_l} y_h^* - \frac{\alpha}{\alpha^2 + \beta_l} \varepsilon_t \quad (18)$$

$$y_t^P(\beta_l, y_l^*) = \frac{\beta_l}{\alpha^2 + \beta_l} \varepsilon_t, \quad (19)$$

and the value of the loss function is

$$L^P(\varepsilon_t) \equiv \frac{1}{2} \left( y_t^D(\beta_h, y_l^*) - y_h^* \right)^2 + \frac{\beta_l}{2} \left( \pi_t^D(\beta_h, y_l^*) \right)^2,$$

which gives

$$L^P(\varepsilon_t) = \frac{1}{2} \varepsilon_t^2 \left( \left( \frac{\beta_l}{\alpha^2 + \beta_l} \right)^2 + \beta_l \left( \frac{\alpha}{\alpha^2 + \beta_l} \right)^2 \right) - \varepsilon_t \left( \frac{\beta_l}{\alpha^2 + \beta_l} y_h^* + \beta_l \frac{\alpha}{\beta_l} \frac{\alpha}{\alpha^2 + \beta_l} y_h^* \right) + \frac{1}{2} (y_h^*)^2 \left( 1 + \beta_l \left( \frac{\alpha}{\beta_l} \right)^2 \right).$$

The expected loss is then

$$\mathbf{E}L^P \equiv \mathbf{E}[L^P(\varepsilon_t)] = \frac{1}{2} \sigma_\varepsilon^2 \frac{\beta_l}{\alpha^2 + \beta_l} + \frac{1}{2} (y_h^*)^2 \frac{\alpha^2 + \beta_l}{\beta_l}.$$

The parameters  $y_l^*$  and  $\beta_h$  do not appear in this formula, because in the punished stage, the central bank plays the discretionary equilibrium. As before, the expected loss rises with the variance of the supply shock  $\varepsilon$ . The fact that  $y_h^*$  is above the natural rate gives rise to the second term.

**Loss when deviating** At the moment when the central bank is deviates, it is facing a representative agent that expects  $\pi_{t+1|t} = \frac{\alpha}{\beta_h} y_l^*$ . In this sense this action cannot happen in equilibrium, because the beliefs of the representative agent are not consistent with the distribution of inflation.

Given these (wrong) inflation expectations, the deviating central banker implements inflation

$$\pi_t^D = \frac{\alpha}{\alpha^2 + \beta_l} \frac{1}{\beta_h} (\beta_h y_h^* + \alpha^2 y_l^* - \beta_h \varepsilon_t) \quad (20)$$

and output

$$y_t^D = \frac{1}{\alpha^2 + \beta_l} \frac{1}{\beta_h} (\alpha^2 \beta_h y_h^* - \alpha^2 \beta_l y_l^* + \beta_l \beta_h \varepsilon_t) \quad (21)$$

It can be shown that the loss would be lower the lower inflation expectations people had.<sup>5</sup> In another words, the more hawkish the central bank pretends to be, the higher is the gain from deviating.

**Loss comparisons** It can be shown that for any value of the supply shock  $\varepsilon$ , the loss is lowest when the central bank deviates, highest when it is punished and intermediate when the central bank behaves according to its announcement and does not deviate;  $L_P(\varepsilon_t) > L_{ND}(\varepsilon_t) > L_D(\varepsilon_t)$ . This situation is depicted in figure 2. The points ND, D and P depict “No Deviation”, “Deviation” and “Punishment” outcomes, corresponding circles capture the value of the central bank’s loss function for  $\varepsilon_t = 0$ .

As was pointed out by the time inconsistency literature, not deviating is not a *one period* equilibrium, which manifests itself in figure 2 by showing that the Phillips curve is not tangent to the iso-loss circles. Furthermore, the loss is smallest under cheating D, but in that case inflation expectations are not consistent with the outcome, so this point is not an equilibrium either.

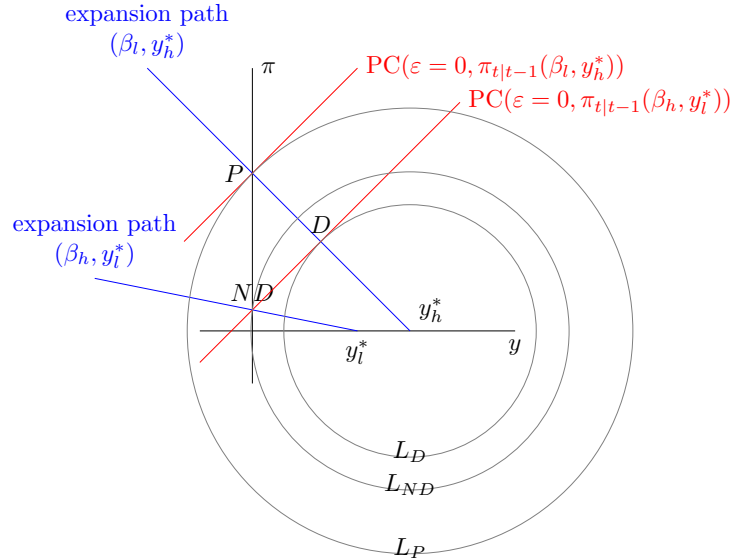


Figure 2: The true preferences  $(\beta_l, y_h^*)$  in the quadratic loss function generate circular (for  $\beta_l = 1$ ) iso-loss curves centered at the bliss point,  $(y, \pi) = (y_h^*, 0)$ . The central bank follows  $(\beta_h, y_l^*)$  instead of its true preferences.

<sup>5</sup>The expected loss when deviating  $EL^D$  is not needed for the value function iteration, hence it is not presented here.



As a useful shortcut, when I say that the central bank behaves according to  $(\beta_h, y_l^*)$ , I mean that the central bank chooses an allocation defined by equations (10) and (11) with  $\beta = \beta_h$  and  $y^* = y_l^*$ . Conversely, when I say that the central bank is punished and behaves according to  $(\beta_l, y_h^*)$ , I mean that it chooses an allocation given by (10) and (11) with  $\beta_l$  and  $y_h^*$ .

### 3.3 Agent's problem

Here I analyze the problem faced by the representative agent: after observing  $\tilde{y}$  and  $\tilde{\pi}$ , she needs to assess the credibility of the central bank in order to set her inflation expectations for the next period. In another words, she has to decide on her beliefs about the conservativeness of the central bank, i.e. whether the central bank really follows the announced  $(\beta_h, y_l^*)$ , or whether it has deviated and behaved according to  $(\beta_l, y_h^*)$ .

In order to do so, the agent has to solve a signal extraction problem. The agent knows that there is noise in the signal and she knows the variances of all shocks, i.e. both supply and noise shocks. The agent knows all the results about the optimal choices of the central bank given its preferences which were derived in the section 3.2.

The representative agent decides only between two possibilities: central bank behaves linearly as if its parameters were  $(\beta_h, y_l^*)$ , or the bank has deviated and it behaved exactly as a discretionary central bank with  $(\beta_l, y_h^*)$ , without any dynamic consideration.<sup>6</sup>

#### 3.3.1 Agent's signal extraction problem

The signal extraction is solved using the following consideration. The representative agent knows which allocation the central bank would choose given any value of the supply shock. Hence if the value of the supply shock was known then the noise shocks could be computed simply as a difference between the allocation chosen by the central bank and the observed allocation. Therefore, for any guess of the supply shock  $\hat{\varepsilon}$ , the agents can infer the noise shocks  $\hat{\xi}_\pi$  and  $\hat{\xi}_y$ . The inferred value of the shock  $\varepsilon$  is then the value which maximizes the joint likelihood function of all the three shocks.

**Assumption:** Shocks  $\varepsilon_t$ ,  $\xi_{\pi,t}$  and  $\xi_{y,t}$  are independently normally distributed with known variances  $\sigma_\varepsilon^2$ ,  $\sigma_\pi^2$  and  $\sigma_y^2$ . ( $\Sigma = \text{diag}(\sigma_\varepsilon^2, \sigma_\pi^2, \sigma_y^2)$ )

As mentioned before, the representative agent either believes the central bank acts according to their announced values or according to their true preference parameters. The advantage of imposing the binary beliefs structure is that the two cases can be solved using the same approach. Let's consider a central bank with generic preferences  $(\beta, y^*)$ , facing some inflation expectations

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<sup>6</sup>In the earlier version of this paper, I implemented a continuous signal extraction, not a binary one which is presented here. The results were similar, but the computation time was significantly higher. The main reason is that there are closed form solutions for the binary problem, whereas additional numerical methods have to be applied in order to solve for the continuous case. This makes the computations of matrix  $P(\varepsilon, d)$  much more demanding, as the signal extraction problem has to be solved both for each point in the grid in the  $(y, \pi)$  space and for each of the simulated noise shocks  $(\xi_y, \xi_\pi)$ . While imposing this binary structure on the beliefs of the representative agent is indeed a simplification, an extra constraint is imposed on the central bank behavior so the assumption about the behavior of the representative agent can be viewed as a reasonable simplification, rather than unrealistic assumption driving the results of this paper. This constraint is discussed in detail in section 3.5.

$\pi_{t|t-1}$  and supply shock  $\varepsilon_t$ :

$$\begin{aligned}\pi_t &= \frac{\alpha}{\alpha^2 + \beta} y^* + \frac{\alpha^2}{\alpha^2 + \beta} \pi_{t|t-1} - \frac{\alpha}{\alpha^2 + \beta} \varepsilon_t \quad \Rightarrow \quad \frac{\partial \pi_t}{\partial \varepsilon_t} = -\frac{\alpha}{\alpha^2 + \beta} \\ y_t &= \frac{\alpha^2}{\alpha^2 + \beta} y^* - \frac{\alpha\beta}{\alpha^2 + \beta} \pi_{t|t-1} + \frac{\beta}{\alpha^2 + \beta} \varepsilon_t \quad \Rightarrow \quad \frac{\partial y_t}{\partial \varepsilon_t} = \frac{\beta}{\alpha^2 + \beta}\end{aligned}$$

The representative agent is aware of the optimal choices described by the two previous equations. The signal extraction is an exercise to obtain some estimates of  $\hat{y}(\tilde{y}, \tilde{\pi})$  and  $\hat{\pi}(\tilde{y}, \tilde{\pi})$ . From equations (3) and (4)

$$\hat{\xi}_{\pi,t}(\tilde{y}, \tilde{\pi}) = \tilde{\pi}_t - \hat{\pi}_t(\tilde{y}, \tilde{\pi}) \quad \Rightarrow \quad \frac{\partial \hat{\xi}_{\pi,t}}{\partial \hat{\varepsilon}_t} = -\frac{\partial \hat{\pi}_t}{\partial \hat{\varepsilon}_t} = \frac{\alpha}{\alpha^2 + \beta} \quad (22)$$

$$\hat{\xi}_{y,t} = \tilde{y}_t - \hat{y}_t(\tilde{y}, \tilde{\pi}) \quad \Rightarrow \quad \frac{\partial \hat{\xi}_{y,t}}{\partial \hat{\varepsilon}_t} = -\frac{\partial \hat{y}_t}{\partial \hat{\varepsilon}_t} = -\frac{\beta}{\alpha^2 + \beta} \quad (23)$$

Now let's use these results for the signal extraction. Using the independence of the shocks, the log-likelihood of a triplet  $(\varepsilon, \xi_y, \xi_\pi)$  for any generic central bank described by  $(\beta, y^*)$  can be written as

$$l(\varepsilon_t, \xi_{t,y}, \xi_{t,\pi}, \beta, y^*) = -\frac{1}{2} \log(2\pi\sigma_\varepsilon^2) - \frac{1}{2} \log(2\pi\sigma_\pi^2) - \frac{1}{2} \log(2\pi\sigma_y^2) - \frac{\varepsilon_t^2}{2\sigma_\varepsilon^2} - \frac{\xi_{\pi,t}^2}{2\sigma_\pi^2} - \frac{\xi_{y,t}^2}{2\sigma_y^2}. \quad (24)$$

Given the inflation expectations  $\pi_{t|t-1}$ , there is a one-to-one mapping  $\varepsilon \rightarrow (y(\varepsilon), \pi(\varepsilon))$  defined in equations (6) and (7), and there is a one-to-one mapping  $((y, \pi), (\tilde{y}, \tilde{\pi})) \rightarrow (\xi_y, \xi_\pi)$  defined by equations (3) and (4). Combining these two results, we can see that the inferred noise can be written as a function of the supply shock,  $(\xi_y, \xi_\pi) = (\xi_y(\varepsilon), \xi_\pi(\varepsilon))$ . The likelihood of the three shocks can thus be rewritten as a likelihood function of  $\varepsilon$  and the observed allocation  $(\tilde{y}, \tilde{\pi})$ :

$$l(\varepsilon_t, \xi_{t,y}, \xi_{t,\pi}, \beta, y^*) \equiv l(\varepsilon_t, \beta, y^*, \tilde{y}, \tilde{\pi}), \quad (25)$$

and since  $l(\varepsilon_t, \beta, y^*, \tilde{y}, \tilde{\pi})$  contains only one unknown variable,  $\varepsilon_t$ , it can be maximized using a simple first order condition.

The agents solve the signal extraction problem by finding the most likely allocation given the observed allocation  $(\tilde{y}, \tilde{\pi}_t)$  using the fact that the noise is the difference between the allocation chosen by the central bank and the observed noisy signal:

$$\begin{aligned}\frac{\partial l(\varepsilon_t, \beta, y^*, \tilde{y}, \tilde{\pi})}{\partial \varepsilon} &= -\frac{\varepsilon_t}{\sigma_\varepsilon^2} - \frac{\xi_{\pi,t}}{\sigma_\pi^2} \frac{\partial \xi_{\pi,t}}{\partial \varepsilon_t} - \frac{\xi_{y,t}}{\sigma_y^2} \frac{\partial \xi_{y,t}}{\partial \varepsilon_t} \\ &= -\frac{\varepsilon_t}{\sigma_\varepsilon^2} - \frac{\xi_{\pi,t}}{\sigma_\pi^2} \frac{\alpha}{\alpha^2 + \beta} + \frac{\xi_{y,t}}{\sigma_y^2} \frac{\beta}{\alpha^2 + \beta} \\ &= -\frac{\varepsilon_t}{\sigma_\varepsilon^2} - \frac{\tilde{\pi}_t - \pi_t}{\sigma_\pi^2} \frac{\alpha}{\alpha^2 + \beta} + \frac{\tilde{y}_t - y_t}{\sigma_y^2} \frac{\beta}{\alpha^2 + \beta}.\end{aligned}$$

Using the allocation computed in (6) and (7), I obtain

$$\begin{aligned} \frac{\partial l(\varepsilon_t, \beta, y^*, \tilde{y}_t, \tilde{\pi}_t)}{\partial \varepsilon} &= -\frac{\varepsilon_t}{\sigma_\varepsilon^2} - \frac{\tilde{\pi}_t - \left( \frac{\alpha}{\alpha^2 + \beta} y^* + \frac{\alpha^2}{\alpha^2 + \beta} \pi_{t|t-1} - \frac{\alpha}{\alpha^2 + \beta} \varepsilon_t \right)}{\sigma_\pi^2} \frac{\alpha}{\alpha^2 + \beta} \\ &+ \frac{\tilde{y}_t - \left( \frac{\alpha^2}{\alpha^2 + \beta} y^* - \frac{\alpha\beta}{\alpha^2 + \beta} \pi_{t|t-1} + \frac{\beta}{\alpha^2 + \beta} \varepsilon_t \right)}{\sigma_y^2} \frac{\beta}{\alpha^2 + \beta} \end{aligned}$$

The first order condition is thus

$$\begin{aligned} 0 &= -\varepsilon_t \left( \frac{1}{\sigma_\varepsilon^2} + \frac{1}{\sigma_\pi^2} \left( \frac{\alpha}{\alpha^2 + \beta} \right)^2 + \frac{1}{\sigma_y^2} \left( \frac{\beta}{\alpha^2 + \beta} \right)^2 \right) + \frac{1}{\alpha^2 + \beta} \left( \frac{\beta}{\sigma_y^2} \tilde{y}_t - \frac{\alpha}{\sigma_\pi^2} \tilde{\pi}_t \right) \\ &+ y^* \left( \frac{\alpha}{\alpha^2 + \beta} \right)^2 \left( \frac{1}{\sigma_\pi^2} - \beta \frac{1}{\sigma_y^2} \right) + \pi_{t|t-1} \frac{\alpha}{(\alpha^2 + \beta)^2} \left( \alpha^2 \frac{1}{\sigma_\pi^2} - \beta^2 \frac{1}{\sigma_y^2} \right), \end{aligned}$$

so the resulting inferred shock  $\hat{\varepsilon}_t$  has to be

$$\hat{\varepsilon}(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t) = \frac{\frac{1}{\alpha^2 + \beta} \left( \frac{\beta}{\sigma_y^2} \tilde{y}_t - \frac{\alpha}{\sigma_\pi^2} \tilde{\pi}_t \right) + y^* \left( \frac{\alpha}{\alpha^2 + \beta} \right)^2 \left( \frac{1}{\sigma_\pi^2} - \beta \frac{1}{\sigma_y^2} \right) + \pi_{t|t-1} \frac{\alpha}{(\alpha^2 + \beta)^2} \left( \alpha^2 \frac{1}{\sigma_\pi^2} - \beta^2 \frac{1}{\sigma_y^2} \right)}{\frac{1}{\sigma_\varepsilon^2} + \frac{1}{\sigma_\pi^2} \left( \frac{\alpha}{\alpha^2 + \beta} \right)^2 + \frac{1}{\sigma_y^2} \left( \frac{\beta}{\alpha^2 + \beta} \right)^2},$$

where the inflation expectations satisfy  $\pi_{t|t-1} = \frac{\alpha}{\beta_h} y_l^*$ , because agents are assumed to believe the announcement at the start of the game.

Now, given  $\hat{\varepsilon}(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t)$ , the inferred allocation chosen by a central bank (following  $(\beta, y^*)$ ) is

$$\hat{\pi}_t(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t) = \frac{\alpha}{\alpha^2 + \beta} y^* + \frac{\alpha^2}{\alpha^2 + \beta} \pi_{t|t-1} - \frac{\alpha}{\alpha^2 + \beta} \hat{\varepsilon}(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t), \quad (26)$$

$$\hat{y}_t(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t) = \frac{\alpha^2}{\alpha^2 + \beta} y^* - \frac{\alpha\beta}{\alpha^2 + \beta} \pi_{t|t-1} + \frac{\beta}{\alpha^2 + \beta} \hat{\varepsilon}(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t), \quad (27)$$

and the noise shocks are

$$\hat{\xi}_\pi(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t) = \tilde{\pi}_t - \hat{\pi}_t(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t), \quad (28)$$

$$\hat{\xi}_y(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t) = \tilde{y}_t - \hat{y}_t(\beta, y^*, \tilde{y}_t, \tilde{\pi}_t). \quad (29)$$

As a shortcut, let's denote the values inferred using  $(\beta_h, y_l^*)$  by subscript  $ND$  (for example  $\hat{\varepsilon}_{ND}(\tilde{y}_t, \tilde{\pi}_t) \equiv \hat{\varepsilon}(\beta_h, y_l^*, \tilde{y}_t, \tilde{\pi}_t)$ ) and the values inferred using  $(\beta_l, y_h^*)$  by subscript  $D$  ( $\hat{\varepsilon}_D(\tilde{y}_t, \tilde{\pi}_t) \equiv \hat{\varepsilon}(\beta_l, y_h^*, \tilde{y}_t, \tilde{\pi}_t)$ ).

Let's denote the log-likelihood that the observed allocation  $(\tilde{y}_t, \tilde{\pi}_t)$  is an outcome of the central bank either *not deviating* or *deviating* by

$$l_{ND}(\tilde{y}_t, \tilde{\pi}_t) = l \left( \hat{\varepsilon}_{ND}(\tilde{y}_t, \tilde{\pi}_t), \hat{\xi}_{\pi, ND}(\tilde{y}_t, \tilde{\pi}_t), \hat{\xi}_{y, ND}(\tilde{y}_t, \tilde{\pi}_t) \right), \quad (30)$$

$$l_D(\tilde{y}_t, \tilde{\pi}_t) = l \left( \hat{\varepsilon}_D(\tilde{y}_t, \tilde{\pi}_t), \hat{\xi}_{\pi, D}(\tilde{y}_t, \tilde{\pi}_t), \hat{\xi}_{y, D}(\tilde{y}_t, \tilde{\pi}_t) \right). \quad (31)$$

### 3.3.2 Comparing log-likelihoods and Revision Region $\mathcal{R}$

The representative agent revises her beliefs about the central bank's trustworthiness if  $l_{ND}(\tilde{y}_t, \tilde{\pi}_t)$  is too low compared to  $l_D(\tilde{y}_t, \tilde{\pi}_t)$ . However, it is not necessarily welfare maximizing to revise the beliefs when the latter likelihood is just marginally bigger, so the representative agent might want to have some cushion in decision making. This is captured by the parameter  $\kappa$ . Let's formally define the *Rejection region*  $\mathcal{R}$ :

$$(\tilde{y}, \tilde{\pi}) \in \mathcal{R} \Leftrightarrow \frac{l_D(\tilde{y}_t, \tilde{\pi}_t)}{l_{ND}(\tilde{y}_t, \tilde{\pi}_t)} < \kappa \quad (32)$$

The parameter  $\kappa$  affects how stringent the agent is when evaluating the deviations. She chooses  $\kappa$  to minimize her own loss. However, because the loss function of the representative agent is the same as the loss of the central bank (disregarding the noise), we can solve for  $\kappa$  as a minimizer of the value function, given that the constraints on the behavior of the central bank are satisfied.

*No rejection region*  $\mathcal{N}$  is the complement region to the *Rejection Region*, i.e. a region such that if an allocation  $x \in \mathcal{N}$  is observed, then the representative agent does not revise her belief, i.e. does not trigger punishment strategy. The region depends on the shape of the likelihood functions and the parameter  $\kappa$ .

## 3.4 Central bank's problem

There are two state variables in the central bank problem; the supply shock  $\varepsilon_t$  and the inflation expectations  $\pi_{t|t-1}$ . There is only one choice variable, the deviation (which can be modelled as either a discrete or a continuous variable, see section 3.5). By assumption, the supply shock  $\varepsilon_t$  is exogenous and not autocorrelated. Hence by construction,  $\varepsilon_t$  is independent of the inflation expectations  $\pi_{t|t-1}$  and the beliefs of the representative agent, as these were formed in time  $t-1$ .

The central bank faces an identical problem every period in which it is not in the punished state. Hence the problem can be formulated in a recursive way. It can be solved using the Bellman equation of the central bank's problem at the beginning of the game. The Optimal policy function is then found by implementing a value function iteration algorithm.

The Bellman equation has form of

$$V(\varepsilon_t, (\beta, y^*)_t) = \min \{L(\varepsilon_t, (\beta, y^*)_t) + \delta \text{EV}(\varepsilon_{t+1}, (\beta, y^*)_{t+1})\}, \quad (33)$$

where  $V(\varepsilon_t, (\beta, y^*)_t)$  is the value function at time  $t$  in the state with supply shock  $\varepsilon_t$  and the beliefs captured by  $(\beta, y^*)_t$ .

The problem of the central bank is to find the optimal policy function  $\text{PF} : (\varepsilon_t, (\beta, y^*)_t) \rightarrow (y_t, \pi_t)$ , which maps the states into allocations. In particular, it specifies what inflation  $\pi(\varepsilon_t, (\beta, y^*)_t)$  and output  $y(\varepsilon_t, (\beta, y^*)_t)$  the central bank chooses given the supply shock  $\varepsilon_t$  and the state of the beliefs  $(\beta, y^*)_t$ . As there are only two values of possible beliefs, and since the behavior in the punished state was given by equation (18) and (19), the problem reduces to solving for the policy function in the not punished state. So the only state variable is in fact  $\varepsilon_t$ .

Here, the supply shock  $\varepsilon$  is exogenous and iid, so out of control of the central bank. However, the beliefs are under control (albeit imperfect), because the decision to stick with  $\pi_{t+1|t} =$

$\pi_{t+1|t}(\beta_h, y_l^*)$  or punish by  $\pi_{t+1|t} = \pi_{t+1|t}(\beta_l, y_h^*)$  depends on the observed allocation  $(\tilde{y}_t, \tilde{\pi}_t)$ , which is a noisy signal about the actual allocation chosen by the central bank. To simplify the notation, I will exploit the fact that the beliefs in period  $t$ ,  $(\beta, y^*)_t$ , can take only two values, either  $(\beta_h, y_l^*)$  in the *not punished* state or  $(\beta_l, y_h^*)$  in the *punished* state, I will denote  $V(\varepsilon_t, (\beta, y^*)_t)$  as

$$V(\varepsilon_t, (\beta, y^*)_t) = \begin{cases} V^P(\varepsilon_t) & \text{if } (\beta, y^*)_t = (\beta_l, y_h^*), \\ V^{NP}(\varepsilon_t) & \text{if } (\beta, y^*)_t = (\beta_h, y_l^*). \end{cases}$$

The policy function can be viewed as generated by a *deviation function*,

$$d(\varepsilon_t, (\beta, y^*)) \rightarrow \mathbb{R}, \quad (34)$$

which assigns a real number to each point in the state space, capturing how much is optimal to deviate from the allocation given by  $(\beta_h, y_l^*)$ :

$$\pi_t(\beta_h, y_l^*, d) = \frac{\alpha}{\beta_h} y_l^* - \frac{\alpha}{\alpha^2 + \beta_h} \varepsilon_t + \frac{1}{\alpha} d, \quad (35)$$

$$y_t(\beta_h, y_l^*, d) = \frac{\beta_h}{\alpha^2 + \beta_h} \varepsilon_t + d. \quad (36)$$

In detail, if the central bank deviates by  $d$ , it chooses an allocation by moving along the Phillips curve. Furthermore, if the central bank implements either full or no deviation, then that does not mean that  $\forall \varepsilon, d(\varepsilon) \in \{0, 1\}$ , but the  $d(\varepsilon)$  is a linear function of the supply shock, as can be seen from figure 2, because the slopes of the two expansion paths are different.

Because I always start from the situation where the central bank announces  $(\beta_h, y_l^*)$  and the announcement is believed, I will write  $\pi_t(\varepsilon_t, d)$  and  $y_t(\varepsilon_t, d)$  as a short cut for  $\pi_t(\beta_h, y_l^*, d)$  and  $y_t(\beta_h, y_l^*, d)$  in situations where this shortcut cannot lead to confusion. Also, in the same way, I sometimes use PF as map between the states and the chosen outcomes  $\pi_t(\varepsilon_t, d)$  and  $y_t(\varepsilon_t, d)$ , meaning  $\text{PF} : (\varepsilon_t, \hat{\beta}_t) \rightarrow (\pi_t(\varepsilon_t, d), y_t(\varepsilon_t, d))$ , where  $\hat{\beta}_t$  captures the state of the beliefs of the representative agent.

I will proceed by focusing on the value functions. There is still one missing ingredient, the description of the law of motion for the beliefs of the representative agent. I will describe this mechanism in section 3.4.2, where I explain how the probability that the representative agent revises her beliefs is affected by the allocation chosen by the central bank.

### 3.4.1 Value function depending of the state of beliefs

**Value in the punished state** Using the notation defined above,  $V^P(\varepsilon)$  denotes the value function when the representative agent no longer believes that  $(\beta_h, y_l^*)$  is the correct description of the central bank actions (=being Punished). The inflation expectations are then set according to  $(\beta_l, y_h^*)$ , which gives  $\pi_{t|t-1} = \frac{\alpha}{\beta_l} y_h^*$ , as derived in equation (9).

In the setting where the punishment period is infinite, the expected value of the loss function is just the discounted value of the expected losses in the punished state, because there will never

be any change in the beliefs. This means that

$$\mathbb{E}[V^P(\varepsilon)] = \mathbb{E} \sum_{t=0}^{\infty} L_t(\beta_l) = \frac{1}{1-\delta} \mathbb{E} L^P. \quad (37)$$

On the other hand, if the punishment is for one period only, then

$$\mathbb{E}[V^P(\varepsilon)] = \mathbb{E} L^P + \delta \mathbb{E} V^{NP}(\varepsilon_t). \quad (38)$$

**Value in the not punished state**  $V^{NP}(\varepsilon)$  labels the value of not being punished facing the supply shock  $\varepsilon$ . This is the problem which the central bank faces at the beginning of the game, and which is being analyzed here.

In the final model, I will allow the central bank to deviate by any amount. However, for the sake of the exposition, let's assume that the deviation can be either none or complete, denoted by either  $ND$  or  $D$ , where  $ND$  means no deviation, and  $D$  means the full deviation to  $(\beta_l, y_h^*)$  outcome. The central bank chooses the action that minimizes its loss. Hence the value of not being punished is the smaller value of either deviating or not deviating:

$$V^{NP}(\varepsilon) = \min\{V_D^{NP}(\varepsilon), V_{ND}^{NP}(\varepsilon)\}, \quad (39)$$

where

$$\begin{aligned} V_D^{NP}(\varepsilon) &= L_D(\varepsilon) + \delta [\mathbb{P}(\varepsilon, D) \mathbb{E}[V^P(\varepsilon')] + (1 - \mathbb{P}(\varepsilon, D)) \mathbb{E}[V^{NP}(\varepsilon')]] \\ V_{ND}^{NP}(\varepsilon) &= L_{ND}(\varepsilon) + \delta [\mathbb{P}(\varepsilon, ND) \mathbb{E}[V^P(\varepsilon')] + (1 - \mathbb{P}(\varepsilon, ND)) \mathbb{E}[V^{NP}(\varepsilon')]], \end{aligned}$$

where  $\mathbb{P}(\varepsilon, X)$  denotes the probability of punishment when facing supply shock  $\varepsilon$  and choosing action  $X \in \{D, ND\}$  and  $\delta$  is the central bank discount factor.

### 3.4.2 Probability of Punishment

$\mathbb{P}(\varepsilon, X)$  is the probability of punishment. Given  $\varepsilon_t$ , the central bank can either decide to stick to the promised strategy and set  $(y_t, \pi_t)$  allocation according to  $(\beta_h, y_l^*)$ , or it can deviate and behave according to  $(\beta_l, y_h^*)$ .

Recall that the *Rejection Region*  $\mathcal{R}$  (formally defined in equation (32)) gives the area in output x inflation space, such that if  $(\tilde{y}, \tilde{\pi}) \in \mathcal{R}$  is observed, the representative agent will decide to punish the central bank and revise their expectations for the next period.

If  $\mathcal{R} = \mathcal{R}(\kappa)$  is the *Rejection Region*, then  $\mathbb{P}(y, \pi)$  can be written formally as

$$\mathbb{P}(y, \pi) = \text{Prob}((y, \pi) + (\xi_y, \xi_\pi) \in \mathcal{R}). \quad (40)$$

The allocation chosen by the central bank is a function of the supply shock and the deviation (as defined in (35) and (36)), so  $\mathbb{P}(y, \pi)$  can be written as  $\mathbb{P}(y_t(\beta_h, y_l^*, d), \pi_t(\beta_h, y_l^*, d))$  and for given announcement  $(\beta_h, y_l^*)$ , it can be written as  $\mathbb{P}(\varepsilon_t, d)$ .

The probability defined in equation (40) can also be written using an indicator function of the

event:  $(y, \pi) + (\xi_y, \xi_\pi) \in \mathcal{R}$ :

$$P(y, \pi) = E_\xi [I_{\{(y, \pi) + (\xi_y, \xi_\pi) \in \mathcal{R}\}}]. \quad (41)$$

The nonlinearity of the policy function is hence the result of the following considerations: even when the central bank never deviates, the likelihood of being punishment is changing in the size of the shock, because the boundaries of the no rejection region are generally not parallel to the announced expansion path (even if  $\beta_h = \beta_l$ ). If it could, the central bank would like to choose to marginally deviate and make the policy tighter to compensate for the change in the likelihood.<sup>7</sup> The intuition is that punishment leads to higher inflation expectations, hence worse inflation-output trade-off and ultimately a higher loss for the central bank. The central bank thus wants to avoid punishment, or at least decrease its likelihood.

### 3.5 Learning and agent's rationality and constraints on central bank's behavior $E[d(\varepsilon)] = 0$

There are two important simplifications in my model that need to be justified. First, the representative agent does not use past observations to form her beliefs. Second, she only considers two possible strategies for the central bank's behavior: full deviation or no deviation at all.

The fully rational representative agent would have access to the full history of allocation and she would use this information to form expectations about the future realization of inflation. In general, such settings are rigorously studied from the game theory perspective. However, even in the game theory, it is common to consider only the so called Markov strategies, which consider only the last realization of the state variable (Mailath and Samuelson, 2006, page 177).

In this paper I use this simplification and assume that the representative agent uses only the current observation when deciding about the beliefs for the next period. Admittedly, it would be more realistic to assume that the representative agent learns from the history. The two most commonly used modes of learning are least squares and bayesian updating. Both approaches necessarily introduce at least one extra state variable.

For simplicity, let's keep the two possible states of inflation expectations,  $(\beta_h, y_h^*)$  and  $(\beta_l, y_l^*)$ , and let's assume that the representative agent updates her beliefs about the credibility  $\mathcal{C}$  of the central bank.  $\mathcal{C}$  fluctuates because of noise or because of deliberate action of the central bank. If the credibility  $\mathcal{C}$  hits some threshold, the representative agent will switch the beliefs from  $(\beta_h, y_h^*)$  to  $(\beta_l, y_l^*)$ .

Any state variable which affects the decisions of the representative agent necessarily enters the Probability of punishment  $P(\varepsilon, d)$ . The problem of the central bank is thus complicated by the fact that today's choice of allocation will affect the probability of punishment tomorrow, even if the beliefs are not rejected today.

This introduces two considerations, first, the policy function of the central bank would not only be a function of true fundamentals (the supply shock  $\varepsilon$ ), but also of the noise, which has affected the updating process of the representative agent.

<sup>7</sup>Note that this deviation goes against the traditional deviation to inflate the economy due to the time inconsistency.

Secondly, adding an extra state variable would result in either increasing the computation time by at least the factor equal to the number of points in the grid for the beliefs tracking variable.<sup>8</sup> Unfortunately, beliefs enter the computations at two places. First, the dimensionality of the probability matrix  $\mathbf{P}$  (implementation is described in section 4.2) would increase from two to three. Furthermore, the extra state variable would also enter the value function iteration increasing the dimensionality from one to two (see section 4.1).<sup>9</sup>

Although computation considerations prevent the implementation of learning in the setting outlined in this paper, clearly it would be wrong to simply ignore it. The outcome of doing so, while maintaining the belief revision process of the representative agent as outlined, would allow the central bank to exploit this irrationality and systematically deviate. In the numerical solution of the model this case is also implemented and serves as another benchmark (see figures 5 and 7 on pages 33 and 35, denoted by the “optimal policy, no constraint” line).

The approach taken here is to impose additional constraints on the central bank and the aim is to get a policy function which would be only a function of the supply shock  $\varepsilon$ . Consider the central bank over a long period of time. Assume that, while it is allowed to fluctuate from one period to the next, the credibility  $\mathcal{C}$  suggests that the central bank behaves exactly according to its announcement  $(\beta_h, y_i^*)$  *on average over a long period of time*. Then it is possible to find an *average policy function*, which would be an average over policy functions given different values of the credibility  $\mathcal{C}$ , weighted by the distribution of  $\mathcal{C}$ . Now assume that this average policy function is such that the average contribution to  $\mathcal{C}$  is zero, weighted by the distribution of the supply shock  $\varepsilon$ . If that is the case, then the initial assumption on long run property of credibility  $\mathcal{C}$  is satisfied by construction.

Now, while I am not able to compute the optimal policy functions for any given value of the credibility variable  $\mathcal{C}$ , I will compute the average policy function. In this sense my model takes the long run behavior of the central bank into account even though learning is not explicitly modeled.

As mentioned above the second simplification in my model is that during the signal extraction exercise, the representative agent considers only two possibilities: the central bank behaves either according to  $(\beta_{high}, y_{low}^*)$  or  $(\beta_{low}, y_{high}^*)$ . This assumption was imposed mainly due to computational aspects. While it is true that this limits the rationality of the representative agent, I am going to argue that it is a less strict assumption than it might appear.

First, in general, the representative agent tests the two hypotheses about the behavior of the central bank. Not allowing the representative agent to use some nonlinear econometric technique indeed limits her rationality. However, I believe this assumption is not too strict as, most of general purpose macroeconomic models feature a linear policy rule.

Second, given the constraint defined later in this section, the linear expansion path given by  $(\beta_h, y_i^*)$  is the best linear description of the behavior of the central bank and it is exactly this expansion path which would be fitted by the representative agent, if she wanted to estimate a

<sup>8</sup>The current computation time using the faculty Intel Duo Core (3.00GHz) PC is about 2 hours to compute the  $\mathbf{P}$  matrix (using 250 draws for the noise pairs  $(\xi_y, \xi_\pi)$ ) and about 1-2 hours for the value function iteration, requiring about three to four hours to solve the model for one parametrization.

<sup>9</sup>The dimensionality problem could be reduced, if the updates in beliefs were not treated as continuous, but only two values would be considered; high or low credibility of the central bank. For an observed allocation which suggests that the central bank has deviated, the representative agent would be more likely to revise her expectations if the credibility was already low and at the same time she would revise the high credibility to low credibility. This setting might capture some of the dynamics. However, even for this setting, the computation time would increase by at least 100%.



linear regression between inflation and output.

The nonlinearity in the policy function arises because the information asymmetries make perfect immediate detection impossible. However, the central bank cannot possibly deviate in a linear fashion, because such behavior would be detectable in the long run, since the condition of the rational expectations of inflation  $E\pi = \pi_{t|t-1}$  would not be satisfied. However, if it is optimal not to deviate at all, the constraint  $E[d(\varepsilon)] = 0$  is trivially satisfied and hence such a policy function is not ruled out.

The constraint  $E[d(\varepsilon)] = 0$  hence seems like a natural way to enforce a deviation function such that the inflation expectations of the representative agent given the announcement  $(\beta_h, y_t^*)$  correspond with the mean of the distribution of realized inflation.

Furthermore, if  $E[d(\varepsilon)] = 0$  is satisfied, then the (linear) expansion path given by  $(\beta_h, y_t^*)$  cannot be rejected against any other linear policy rule. Hence, this restriction and the fact that there is no persistence in the supply shock justify the assumption that the agents do not use any past observations for inference about the current action of the central bank.

## 4 Implementation

Since there is no closed form solution to the central bank's problem, the model has to be solved numerically. In this section I describe the implementation approach I used. In particular, I focus on the way I implement the value function iteration and how I solve for the probability of punishment for any given allocation choice of the central bank. The matlab files are available at [people.pwf.cam.ac.uk/fr282/1st\\_year\\_paper.zip](http://people.pwf.cam.ac.uk/fr282/1st_year_paper.zip)

### 4.1 Implementing the Value function iteration algorithm with a constraint

The standard value function algorithm converges point-wise, i.e. the choice of optimal policy at any point of the state space does not depend in any way on the choice at any other point of the state space.

However, imposing  $E[d(\varepsilon)] = 0$  is a restriction on all the points jointly and hence requires an adjustment in the value function iteration algorithm. The reason is that choosing  $d(\varepsilon_1)$  marginally higher, requires, that on average (appropriately weighted)  $d(\varepsilon)$  has to be lower for all other  $\varepsilon$  in the state space.

I proceed in the following way. The variable being minimized is not the loss at each point of the state space, but the *expected value* of the value function over all grid points jointly. Originally, the policy function is an infinite dimensional object which maps each point in the state space to a particular value of the deviation. Here, the policy function is approximated and the optimization runs over much fewer coefficients of this approximation. I use two possible approximation techniques: chebyshev polynomials and piece-wise linear approximation.

Formally, recall that the policy function  $PF$  maps shocks  $\varepsilon$  into  $d$ , deviations from the expansion path defined by  $(\beta_h, y_t^*)$  in equation (12). Assume that  $PF(\varepsilon)$  can be approximated by some

function of parameters  $\boldsymbol{\theta}$ ,  $\mathcal{P}_{\boldsymbol{\theta}}(\varepsilon)$ . Also recall, that the value function in the not-punished state is

$$V^{NP}(\varepsilon_t) = \min_{d_t} \{L(\varepsilon_t, d_t) + \delta [P(\varepsilon_t, d_t)EV^{NP} + (1 - P(\varepsilon_t, d_t))EV^P]\}. \quad (42)$$

The problem is that  $d(\varepsilon_t)$  is in theory an infinite dimensional object and even with discretization, the grid is still too fine to solve for it. However, if we use the approximation of the policy function  $\mathcal{P}_{\boldsymbol{\theta}}(\varepsilon) \approx d(\varepsilon_t)$ , which is a function of some parameters  $\boldsymbol{\theta}$ , we get

$$V^{NP}(\varepsilon_t) = \min_{\boldsymbol{\theta}} \{L(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon)) + \delta [P(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon))EV^{NP} + (1 - P(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon)))EV^P]\}, \quad (43)$$

where  $\boldsymbol{\theta} = \{\theta_i\}_{i=0}^{n-1}$  might be the parameters defining an approximation of the policy function by a polynomial  $\mathcal{P}_{\boldsymbol{\theta}}(\varepsilon) = \sum_{i=0}^{n-1} \theta_i \varepsilon_t^i$ , or any other approximation of the policy function. The important fact is that  $\boldsymbol{\theta}$  is a relatively small vector, in particular of an order of magnitude smaller than the grid for the state variable  $\varepsilon$ .

Furthermore, some mechanism which links the choices for all the values of the state variable is needed. The natural choice is to minimize *the expected value* of the value function,  $EV^{NP}(\varepsilon_t)$ . The expected value is approximated numerically by using quadrature. This setting can be directly implemented using Matlab's nonlinear minimization procedure `fmincon`, which incorporates the nonlinear constraint.

However, the drawback is that for very high values of  $|\varepsilon|$ , the likelihood of being in this state is very low. Therefore the change of  $EV(\varepsilon)$  given a marginal change in  $\boldsymbol{\theta}$  is getting smaller as  $|\varepsilon|$  gets bigger. The implication is that the numerical solution is more precise for smaller values of  $|\varepsilon|$ .

**Implementation summary** To sum up, in each step of the value function iteration, the problem is to find parameters  $\boldsymbol{\theta} = \{\theta_i\}_{i=0}^{n-1}$  of some choice of approximation function, which minimizes the expected value of the value function. Formally,

$$\begin{aligned} & \text{find } \boldsymbol{\theta} \text{ to minimize} \\ & \mathbb{E}[V^{NP}(\varepsilon_t)] = \mathbb{E}\left[L(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon)) + \delta \left[P(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon))EV^{NP} + (1 - P(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon)))EV^P\right]\right] \\ & \text{such that } \mathbb{E}[\mathcal{P}_{\boldsymbol{\theta}}(\varepsilon)] = 0. \end{aligned}$$

Assuming that there are initial guesses for  $EV^{NP}$  and  $EV^P$ , the algorithm is the following:

1. run the minimization sub-routine:
  - (a) construct the probability of rejection  $P(\varepsilon, d) = P(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon))$  by interpolation of  $P(y, \pi)$  matrix as a function of  $\varepsilon$  and  $\boldsymbol{\theta}$
  - (b) construct the value function as a function of  $\varepsilon$  and  $\boldsymbol{\theta}$  using the interpolated matrix  $P(\varepsilon_t, \mathcal{P}_{\boldsymbol{\theta}}(\varepsilon))$
  - (c) construct the expected value of the value function as a function of  $\boldsymbol{\theta}$  using some numerical integration method,  $EV(\varepsilon, \boldsymbol{\theta}) = f(\boldsymbol{\theta})$

- (d) construct  $E[d(\varepsilon)] = g(\boldsymbol{\theta})$  as a function of coefficients in the policy function approximation
  - (e) run `fmincon` on  $f(\boldsymbol{\theta})$  using  $g(\boldsymbol{\theta}) = 0$  as a constraint, obtaining a new value function  $V^{NP}$
2. update  $EV^{NP}$  and then  $EV^P = EL^P + \delta EV^{NP}$
  3. check if the difference in  $EV^{NP}$  is small, if so, stop, if not start again

## 4.2 P matrix

Let  $N_y$  and  $N_\pi$  be the number of grid points for output and inflation, then the  $N_y \times N_\pi$  matrix  $\mathbf{P}$  is the matrix which contains the probability of belief revision for each allocation  $(y, \pi)$  the central bank might choose.

This matrix is constructed in the following way:

1.  $N_{sim}$  pairs of noise shocks are randomly drawn from the normal distribution,
2. for each point in the  $(y \times \pi)$  grid,
  - (a)  $N_{sim}$  observed outcomes  $(\tilde{y}, \tilde{\pi})$  are obtained by adding the noise shocks to the corresponding values of the grid point
  - (b) for each of these  $N_{sim}$  observed outcomes the representative agent solves the signal extraction problem and decides if she keeps or revises her belief
3. the probability of beliefs rejection is then approximated the ratio of the number of observed outcomes where the beliefs are revised over total number of simulated outcomes,  $N_{sim}$ .

I am using grid for  $y$ :(-16:0.25:15), grid for  $\pi$ :(-10:0.25:15) and grid for the supply shock  $\varepsilon$ :(-14:0.05:14). The number of noise shock simulation is  $N_{sim} = 250$ . The resulting probability is very steep, which might suggest that more then 250 simulation are needed.

## 5 Parameter values

In this section I describe the choices of parameter values that I made. I will also explain in detail how I derived these values in order to justify my choices.

The values of  $(\beta_h, y_l^*)$  and  $\kappa$  can be found by a grid search conditional on the rest of the parameters as the values that minimize the expected value of the central bank's (and hence also the social) value of the problem. The discount factor  $\delta$  is assumed to be 0.99. The estimated  $\alpha = 0.9$  by obtained by estimating the Phillips curve from consumer inflation expectations and HP filtered gap measure of output. The socially optimal level of output (gap)  $y_h^* = 0.04$  is obtained from using a textbook new keynesian approach by assuming monopolistic producers and price stickiness. The relative variance of the shocks  $(\sigma_\varepsilon, \sigma_y, \sigma_\pi) = (2.5, 1, 0.2)$  is obtained by exploring the difference between the first release of data and the data revised after one year. Finally, empirical literature is consulted to get an insight what the ration between  $\beta_h$  (set to 8) and  $\beta_l$  (set to 1) in Rogoff's conservative central banker setting.

## 5.1 Slope of the Phillips curve $\alpha$

The Phillips curve in the model is defined as

$$y_t - y_t^{POT} = \alpha(\pi_t - \pi_{t|t-1}) + \varepsilon_t, \quad (44)$$

where  $\pi_{t|t-1}$  is the expectation about period  $t$  inflation formed at time  $t-1$ , the potential output  $y_t^{POT}$  is normalized to zero. This PC is different to standard NK (hybrid) PC, as investigated for example by (Galí and Gertler, 1999), because of the timing of the expectations. The results from the standard literature are therefore not directly applicable and I estimate the parameter myself.

### 5.1.1 Estimation

To capture  $\pi_{t|t-1}$  I use the inflation expectation time series<sup>10</sup> compiled by the University of Michigan obtained from the database FRED.<sup>11</sup> To obtain output gap  $y_t$  I apply the HP filter with smoothing parameter  $\lambda = 1600$  on the logarithm of real GDP.<sup>12</sup> Finally, for inflation<sup>13</sup>  $\pi_t$  I use the year-on-year percentage change.

Equation (44) cannot be estimated as it is. The reason is that in my model, there is no persistence in the variables, whereas the data is strongly persistent. In order to eliminate this issue, I estimate the expectations augmented Phillips curve with additional lagged output gaps as regressors. Two lags of  $y$  on the right hand side of the equation are needed in order not to reject  $H_0$  of no autocorrelation among the fitted residuals. However, due to the strong persistence in the forecast error ( $\pi_t - \pi_{t|t-1}$ ), which is assumed away in my model, I also add lagged inflation expectation error and estimate:

$$y_t = \alpha_1(\pi_t - \pi_{t|t-1}) + \alpha_2(\pi_{t-1} - \pi_{t-1|t-2}) + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t \quad (45)$$

The estimated parameters  $\beta_1$  and  $\beta_2$  do not change after adding the extra lagged inflation expectation error, but the lagged inflation forecast error is significant.

parameter	Coefficient	Std. Error	t-Statistic	Prob.
$\beta_1$	1.152	0.091	12.708	0.0000
$\beta_2$	-0.356	0.091	-3.912	0.0002
$\alpha_1$	0.904	0.286	3.162	0.0020
$\alpha_2$	-0.658	0.274	-2.403	0.0180

Table 1: Estimation of the slope of the Phillips curve

The contemporaneous effect is then measured to be  $\alpha \approx 0.9$ . Hence, in my model I use  $\alpha = 0.9$ . Some other estimates from the literature might suggest lower values, for example Ireland (1999, page 289) estimated ( $\alpha = 0.15$ ). However, Ireland's results are not directly applicable here, as in

<sup>10</sup><http://research.stlouisfed.org/fred2/series/MICH>

<sup>11</sup>Alternative would be to use the survey of professional forecasters, <http://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/>, Mankiw et al. (2003) document significant differences between the two time series.

<sup>12</sup><http://research.stlouisfed.org/fred2/series/GDP>

<sup>13</sup><http://research.stlouisfed.org/fred2/series/CPIAUCSL>

his setting the central bank acts before observing the shock, not afterwards as in my model, and he considers the unemployment, not the output gap.

## 5.2 Signal to noise ratio

Signal to noise ratio is an important factor for the signal extraction problem. To get an idea about the relative magnitudes, I take data for CPI and GDP from FRED database, get year on year growth and inflation and compare different vintage time series. I define noise at time  $t$  as the difference between the just released number and the revised number one year afterwards.

As can be seen from figure 3, the GDP data is much more noisy than the CPI data. The estimated standard deviations are listed in table 2.

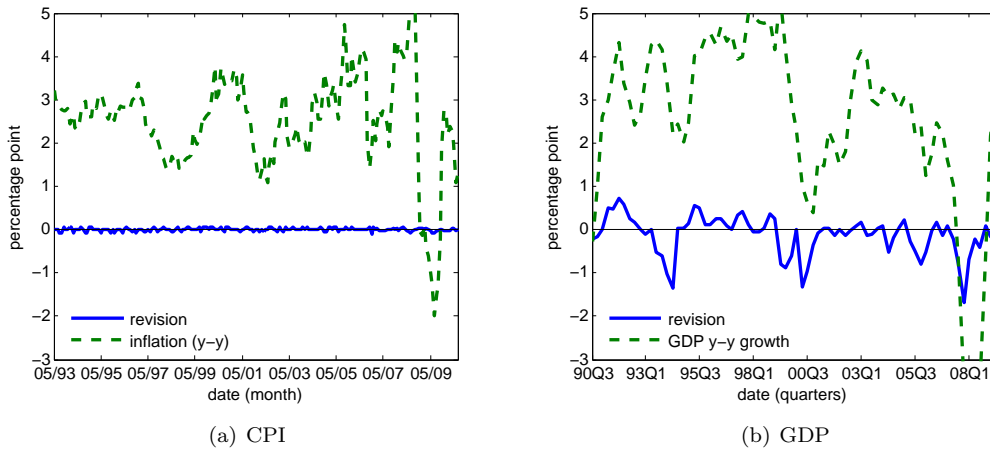


Figure 3: Noise and signal in the data

	GDP	CPI
std of signal	2.09	1.14
std of the noise	0.46	0.04
Signal to noise ratio	4.57	29.16

Table 2: Signal and Noise for CPI and GDP

Because in the signal extraction, it is the relative variance what counts, I scale up the volatility of all shocks. Admittedly, let the variance of the noise to be slightly bigger than what I have found from the data and I use as relative variances of the shocks  $(\sigma_\varepsilon, \sigma_y, \sigma_\pi) = (2.5, 1, 0.2)$ .

## 5.3 Preferences of the central bank

### 5.3.1 Socially optimal level of output $y_h^*$

Galí (2008, page 48) shows that the natural level of output  $y^n$  of a standard New Keynesian model with Calvo pricing can be derived to be

$$y_t^n = \frac{1 + \phi}{\sigma(1 - \alpha) + \phi + \alpha} a_t - \frac{(1 - \alpha)(\mu - \log(1 - \alpha))}{\sigma(1 - \alpha) + \phi + \alpha},$$

where  $\sigma$  denotes risk aversion,  $\phi$  is the labor supply elasticity,  $\alpha$  is the production function parameter,  $\mu$  is the logarithm of the flexible price mark-up, equal to  $\frac{\epsilon}{\epsilon-1}$ , where  $\epsilon$  is the price elasticity of demand.  $a_t$  is the percentage deviation of the technology from its steady state. In my model,  $y_h^*$  is the analogue of Galí's  $y^n$ .

The simplest possible parametrization is the following:  $\alpha = 0.36$ ,  $\sigma = \phi = 1$ . A conservative value for the desired mark-up is about 20%, which gives  $\mu = 0.2$  (Ireland, 2001, page 10).

The equilibrium under perfect competition would be higher by  $\frac{(1-\alpha)\mu}{\sigma(1-\alpha)+\phi+\alpha}$  percent, which is about 6.4% given the parameter values discussed in the previous paragraph. However, the true risk aversion is probably larger than one and also  $\phi$  might be higher in reality, resulting in higher denominator. Hence, I assume that the social optimal is 4% above the potential level of output. Because the potential level is normalized to zero, this means that  $y_h^* = 4\%$ .

### 5.3.2 Rogoff case: $\beta_h > \beta_l, y_l^* = y_h^*$

Here I try to find values for  $\beta_h$  and  $\beta_l$  which are consistent with each other in the Rogoff setting of a conservative central banker, so  $y_h^* = y_l^* = y^*$ . First, I will look for empirical evidence on  $\beta_h$  and then, given the choice of  $\beta_h$ , I will try to find plausible values for  $\beta_l$ . This parametrization is then used to find the optimal policy and show that this policy is nonlinear even in the setting conjectured by Rogoff.

**Choosing  $\beta_h$**  The standard approach for estimating the preferences of a central bank is to set up a system of equations and estimate them simultaneously using ML or by GMM. The empirical results in the literature are inconclusive. Some authors report that the weight on inflation stabilization is about 8-10 times higher than the one of the output, as for example in Givens (2009) and Favero and Rovelli (2003). However, there is also a paper arguing that the ration is about 1.5 (Ozlale, 2003). Table 4 on page 42 summarizes some evidence on the central bank inflation-output gap trade-off preference parameter.

I set  $\beta_h = 8$ , i.e. the central bank puts 8 times more weight on stabilizing deviations of inflation from its steady state than on stabilizing the output gap.

**Choosing  $\beta_l$**  Now, given  $\beta_h$ , what is a reasonable value for  $\beta_l$ ? A higher value of  $\beta$  decreases the inflation bias, but at the same time the inflation-output gap trade off becomes less favorable. The optimal value can be found using the first order condition on the expected loss  $L_t(\beta_l, \beta_h, \varepsilon_t)$ , a loss of a central bank with preferences described by  $\beta_l$ , which is acting as if it had preferences  $\beta_h$  and is facing shock  $\varepsilon_t$ :

$$\begin{aligned} L_t(\beta_l, \beta_h, \varepsilon_t) &= \frac{1}{2} \left( \frac{\alpha}{\beta_h} y^* - \frac{\alpha}{\alpha^2 + \beta_h} \varepsilon_t - y^* \right)^2 + \frac{\beta_l}{2} \left( \frac{\beta_h}{\alpha^2 + \beta_h} \varepsilon_t \right)^2 \\ \mathbf{E}L_t(\beta_l, \beta_h) &= \frac{1}{2} (\beta_h^2 + \alpha^2 \beta_l) \left( \frac{\sigma_\varepsilon^2}{(\alpha^2 + \beta_h)^2} + \frac{(y^*)^2}{\beta_h^2} \right) \\ \frac{\partial \mathbf{E}L_t(\beta_l, \beta_h)}{\partial \beta_h} &= \alpha^2 \left( \frac{\sigma_\varepsilon^2 (\beta_h - \beta_l)}{(\alpha^2 + \beta_h)^3} - \frac{(y^*)^2}{\beta_h^3} \beta_l \right) \end{aligned} \quad (46)$$

For given  $\beta_l$  (social preferences), putting  $\frac{\partial \mathbf{E}L_t(\beta_l, \beta_h)}{\partial \beta_h}$  solved in (46) equal to zero creates a nonlinear equation which can be solved numerically to obtain the optimal  $\beta_h$  (optimal “pretended”

conservativeness of the central banker). Since I have already determined the optimal degree of conservativeness ( $\beta_h$ ), I can invert the equation to see, what values of  $\beta_h$  imply for possible values of  $\beta_l$ . I am using the values of parameters from above.

The results are depicted at Figure 4. The first two panels show  $\frac{\partial EL_t(\beta_l, \beta_h)}{\partial \beta_h}$  as a function of  $\beta_h$  for different values of  $\sigma_\varepsilon$  and  $\beta_l$ . One can observe that the value of the derivative is negative initially and then positive. The third panel then shows the corresponding value of the expected loss minimizing value of  $\beta_h$  for any given  $\beta_l$ . This information is then used to motivate the choice of  $\beta_h$  in Rogoff setting.

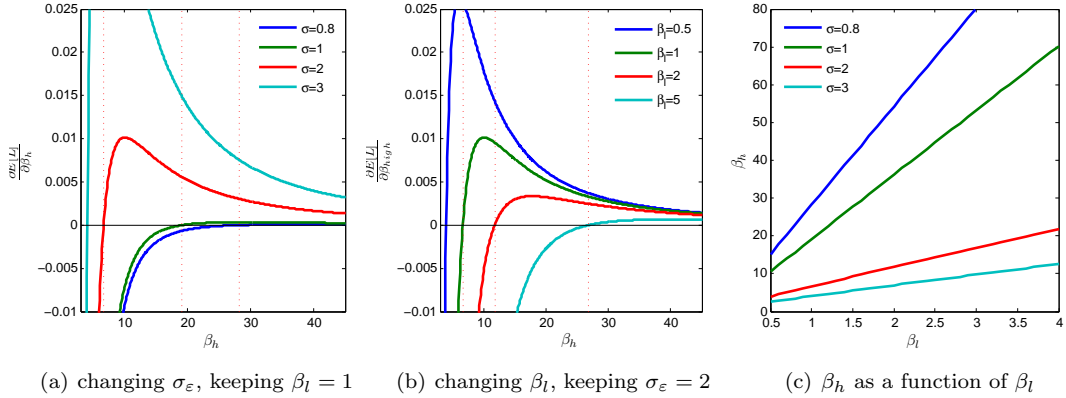


Figure 4: The relation between  $\beta_h$  and  $\beta_l$  in Rogoff's conservative banker setting.

In particular, if the variance of the supply shock is equal to 2, we can see that  $\beta_l$  being between 1 and 2 implies  $\beta_h \approx 8 - 12$ . This is very close to some of the empirical results mentioned above (furthermore, the slope of the expansion path is the inverse of the parameter value, which makes the differences even smaller) and hence I claim that after calibrating the volatility of the supply shock to  $\sigma_\varepsilon = 2.5$ , the other parameter choices are reasonable.

Taylor (1979) argued that while there is no long run trade-off between inflation and output, there is a *second order trade-off*, where the long run volatility of output can be decreased only at the cost of increasing the volatility of inflation. Given the estimates of variance reported in table 2, the long run variance of inflation is about 4 times lower than the variance of output. Considering that the slope of the Phillips curve is about 1, this would suggest that  $\beta_h$  which the central bank actually follows should be around 4 as well. This fact might suggest that, in the Rogoff case, setting  $\beta_h = 8$  with  $\beta_l = 1$  might be at the more conservative end of the interval of the plausible specifications.

## 5.4 Parametrization summary

All the numerical results to be presented in the next sections are computed using the following parameter values:

	$\delta$	$\beta_l$	$\beta_h$	$\alpha$	$y^*$	$\sigma_\varepsilon$	$\sigma_\pi$	$\sigma_y$	$\kappa$
parameters	0.99	1		0.9		2.5	0.2	1	0.8
Blinder			1		0				
Rogoff			8		4				

Table 3: Parametrization

## 6 Results

In order to see the effects of the information asymmetries introduced in the model, I analyze the optimal policy if the central bank announces the Blinder setting ( $\beta_h = \beta_l, y_l^* = 0$ ) and the Rogoff setting ( $\beta_h > \beta_l, y_h^* = y_l^*$ ). As we will see, these announcements will not be optimal in the setting considered in this paper.

The results are presented by showing two-panel figures. Panel (a) shows the policy functions, while panel (b) depicts the implied allocations of output and inflation. Each panel always contains results corresponding to four different regimes:

1. *announced policy* captures what the behavior would be if there was no deviation from the announced policy (black dashed line),
2. *true preferences* shows what the result would be if the central bank chose the allocation according to its true preferences and hence fully deviated by doing so (magenta dashed line),
3. *optimal policy, no constraint* depicts the central bank's optimal policy given the information structure but without implementing the constraint  $\mathbf{E}[d(\varepsilon) = 0]$  (blue line),
4. *optimal policy, with constraint* depicts the optimal policy under the implementing the constraint  $\mathbf{E}[d(\varepsilon)] = 0$ .

Furthermore, each panel contains information about the area between 2.5th and 97.5th percentile of  $\varepsilon$ ; the panels (a) contain vertical dashed lines and the allocation corresponding to either quantile is depicted by a dot on the respective line in the panels (b). Finally, the allocations panels (b) also capture the no rejection region by a green area.

### 6.1 Blinder proposition

Blinder's suggestion was that the central bank in fact targets the natural level of output rather than the socially optimal level. In my setting, this translates to the announcement of  $(\beta_h, y_l^*) = (1, 0)$ . The resulting behavior under these announcements is shown in figure 5.

The optimal policy is more accommodative for positive shocks and less so for negative shocks. This comes from the shape of the no revision region. Contrary to what one might anticipate and despite appearing almost linear in panel (b) of figure 5, in the situation where  $\beta_l = \beta_h$  the border of *No revision region* is not a completely straight line. To see this, consider the signal extraction problem of the representative agent. The agent analyzes two possible actions which could have led to the observed outcome. Assume that the observed allocation is  $(\tilde{y}, \tilde{\pi})$  and that it can be explained in the no deviation case by  $\hat{\varepsilon}_{ND}$  and in deviation case by  $\hat{\varepsilon}_D$ . If the central bank deviates, it is increasing both inflation and output. The higher values of the supply shock increase



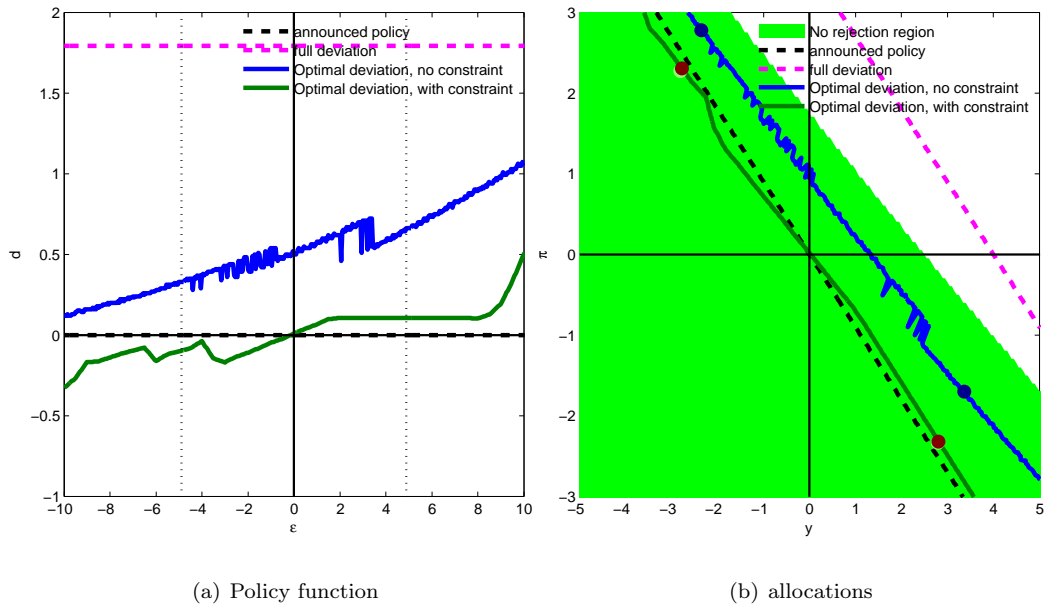


Figure 5: Blinder proposition:  $(\beta_h = \beta_l, y_l^* = 0)$

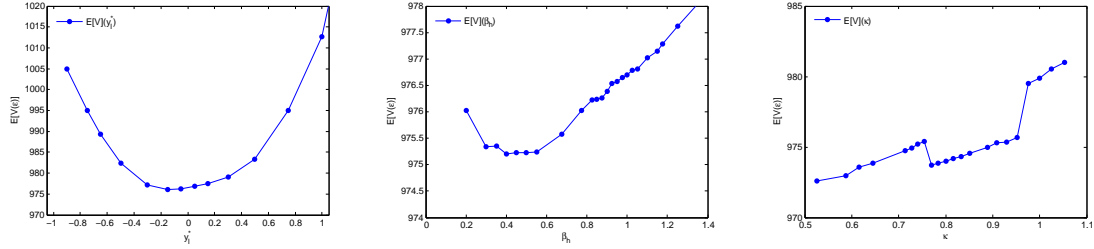
output and decrease inflation and  $\hat{\varepsilon}_D < \hat{\varepsilon}_{ND}$  and this affects the likelihoods. The interplay of this effect with the effect of deviation on inflation ultimately causes the nonlinear shape of the border.

The line connecting the optimal allocations is flatter in the neighborhood of  $\varepsilon = 0$ , as it is documented in panel (b) of figure 5. For slightly larger positive shocks, the central bank deviates positively. However, the magnitude of the deviation is decreasing relative to the size of the shock, so in this sense it is fair to say that the central bank almost follows its announcement. The policy function for negative shocks is not as smooth, but on average the relative size of the deviation decreases with the size of the shock.

For very large shocks, the policy function diverges. However, this is an artifact of the numerical implementation; as the size of the supply shock  $\varepsilon$  gets bigger, the weight of the corresponding outcome in the  $E[V(\varepsilon)]$  falls exponentially. Hence the *expected* value function iteration does not have enough traction in such regions.

The lower slope for small  $\varepsilon$  and higher slope for intermediate  $\varepsilon$  could be interpreted as if the central bank was very hawkish to small shocks, but it would accommodate larger shocks. However, the magnitude of nonlinearity is not large.

Figure 6 shows the expected value of the value function, i.e. the expected value of the loss function given the optimal policy response to  $\varepsilon$  for different values of parameters  $\beta_h, y_l^*$  and  $\kappa$ , starting from the case when the central bank targets the natural level of output and keeping the original trade-off parameter,  $(\beta_h, y_l^*) = (1, 0)$ . The main message of figure 6 is that the society can attain a lower value of the value function in expectations, if the central banker announces a less conservative strategy. The fact that Blinder suggestion is not welfare maximizing was pointed out already by Nobay and Peel (2003). In my setting, the central bank can decrease the expected loss by targeting even smaller output (panel (a) of figure 6), or become less conservative by choosing  $\beta_h$  actually below 1, i.e.  $\beta_h < 1 = \beta_l$ , (panel (b)).



(a) keeping  $(\beta_h, \kappa) = (1, 0.8)$ , changing  $y_i^*$  (b) keeping  $(\kappa, y_i^*) = (0.8, 0)$ , changing  $\beta_h$  (c) keeping  $(\beta_h, y_i^*) = (1, 0)$ , changing  $\kappa$

Figure 6: Value of the  $E[V]$  for different values of parameters, the Blinder case (baseline:  $\beta_h = \beta_l, y_i^* = 0$ ). Each point represents a solved model, the lines are formed by linear interpolation.

There does not seem to be an optimal choice for  $\kappa$  in the interval that was searched. Obviously, such a point might not exist, because as  $\kappa$  get smaller, the agents are less likely to punish the central bank, so the central bank can deviate more and by doing so increase the social welfare. However, the constraint  $Ed = 0$  should prevent the central bank from deviating further and further and so it is necessary to impose an assumption that the value of the cushion parameter  $\kappa$  cannot be below a certain point  $\underline{\kappa} = 0.8$ .

## 6.2 Rogoff proposition

Rogoff (1985) suggested that the inflationary bias can be reduced by introducing a conservative central banker. In my setting, this corresponds to the central bank announcing  $(\beta_h, y_i^*) = (8, 4)$  (the choice these numbers was discussed in section 5.3.2). Any  $\beta_h \in (4, 16)$  would deliver very similar policy functions. The results can be seen in the Figure 7.

The central bank deviates positively for negative values of shocks and negatively for positive shocks. This means that the direction of the deviations/asymmetry is reversed compared to Blinder case. The natural explanation would be following. The motivation for such a behavior stems from the quadratic loss function; it is *relatively* more beneficial to deviate when facing a very negative shock, than when facing a positive shock, so the central bank finds it optimal to be more accommodative in recessions and pay the price of being more hawkish in the booms. However, this cannot be the right intuition, because a similar trade-off is attainable in the Blinder case, but there the central bank does not use this opportunity.

Panel (b) of figure 7 also provide an insight into the interaction of the optimal policy with the no rejection region. Ignoring the constraint  $E[d(\varepsilon)] = 0$  for the moment, the central bank would prefer to deviate towards its true preferences. However, getting closer to the edge of the Rejection region increases probability of beliefs revision and punishment. Given the relative variance of the noise shocks, the central bank is more constrained by the distance of the chosen allocation along  $y$  axis. This effect is clear from the optimal allocation; for  $\varepsilon \in (2, 8)$ , the optimal inflation is inelastic to the changes in the supply shock. The reason is that the allocation is at the boundary of this “safe” zone within the no rejection region. This observation also explains the relatively small magnitude of the nonlinearity in Blinder setting, where the optimal policy was found to be far from the boundary of the no rejection region.

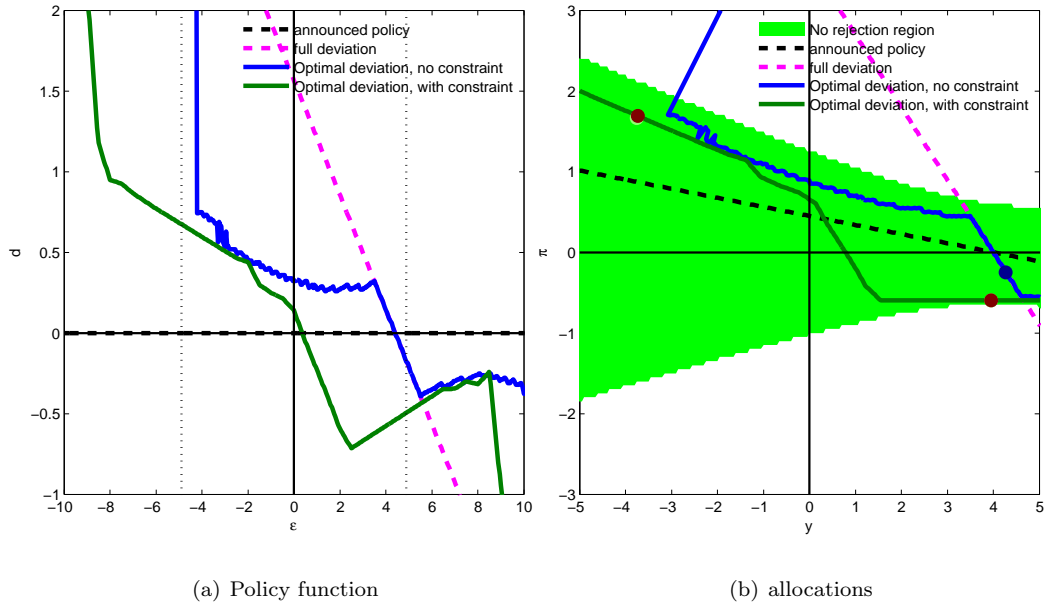


Figure 7: Rogoff proposition:  $(\beta_h > \beta_l, y_l^* = y_h^*)$

The true cause of the strong nonlinearity is the widening gap between the allocation dictated by the true preferences (magenta dashed line) and the announced policy (black dashed line). This gap is bigger the more negative the supply shock is. The central bank hence deviates to decrease this loss. The central bank hits both the lower and the upper boundary of the no revision region  $\mathcal{N}$ . The lower bound is hit for very positive shocks.

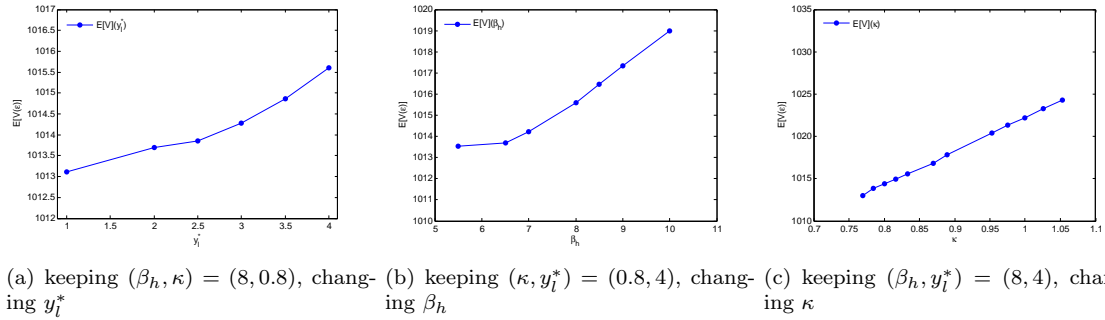


Figure 8: Value of the  $E[V]$  for different values of parameters, the Rogoff case (baseline:  $\beta_h = 8, y_l^* = y_l^* = 4$ ). Each point represents a solved model, the lines are formed by linear interpolation.

Finally, figure 8 shows that in the setting outlined in the model, the Rogoff conservative central banker is also not the optimal approach. First, the value of the expected value function is above the one in the Blinder case which implies lower social welfare. Second, there are no minimizers  $\beta_h$  and  $y_l^*$  of the expected value. Third, the fact that there is no clear minimizing value of the parameter  $\kappa$  is the same as in the Blinder case.

### 6.3 Optimal announcement

The optimal announcement is the pair  $(\beta_h, y_l^*)$  which minimizes the expected loss of the central bank given the value of  $\kappa$ . Because the central bank has the same loss function as the representative agent, this is also the socially optimal policy.

Given the results presented for the Blinder case, it is reasonable, that the optimal announcement would be somewhere close to  $(\beta_h, y_l^*) = (0.5, -0.1)$ . However, these results were only partial in the sense that one of the two variables was always fixed. To find the optimal announcement, it is necessary to run a complete grid search over  $(\beta_h, y_l^*)$ . The problem is that even a very rough grid of  $(\beta_h, y_l^*) \in (0.4, 1.5) \times (-0.3, 0.2)$  with step of 0.1 would require about 60 runs. There is possibility of using some more sophisticated method to choose the grid points more efficiently, as suggested for example by Grune and Semmler (2004), or so called sparse grid methods.

The basic intuition, however, can be given even without obtaining the global minimum. In the situation where  $y^* < 0$ , it might be optimal to have a  $\beta_h < \beta_l$ . In such a situation, the very low value of  $\beta_h$  increases the magnitude of the bias, however, because  $y_l^* < 0$ , the bias is actually negative. Consequently, in equilibrium, the inflation expectations are negative. This makes deviations more profitable. However, at the same time the constraint  $E[d(\varepsilon)]$  pulls the central bank from its true preferences, which counterbalances the potential gain from deviations.

## 7 Robustness checks and sensitivity analysis

In this section, I will discuss the choice of value for parameter  $\kappa$  and the differences arising due to different approximation and numerical integration methods.

### 7.1 The value of the cushion parameter $\kappa$

The search over plausible values of the cushion parameter  $\kappa$  did not discover any clear candidate for the minimizer of the expected value of the value function neither in Blinder's nor Rogoff's setting. I chose the value  $\kappa = 0.83$ , because it seems to be some local minimizer and the value is reasonably close to 1. Figure 9 shows the policy functions over a broad range of values,  $\kappa \in \{0.71, 0.77, 0.83, 0.9, 1\}$ . The two panels show the differences between the linear and the Chebyshev polynomial approximation.

Despite some differences, the general trajectory of the all the policy functions seem to be the same; there are negative deviations for negative shocks and positive deviations for positive shocks.

The parameter  $\kappa$  affects the size of the *No rejection region*  $\mathcal{N}$ . However, it turns out that the effects are not very strong and the regions do not differ much for very different values of  $\kappa$ . This is shown in the figure 10. The differences are very small for the Blinder case. In the Rogoff case the effect of changing  $\kappa$  is slightly stronger on the lower boundary of the  $\mathcal{N}$  region, the behavior at the upper boundary is again very steep.

The underlying reason is the very steep shape of the likelihood functions. The Probability  $P(y, \pi)$  inherits this characteristic. Translated to the language of the model, this means that for very small deviations the central bank faces a very low marginal increase in the probability of punishment. However, for a deviation bigger than a certain threshold, this probability increases significantly and it gets close to 1 very quickly.

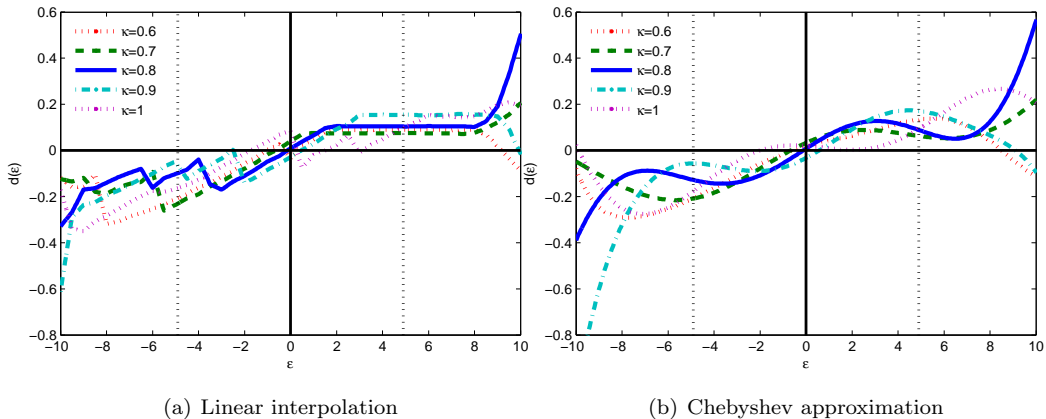


Figure 9: Comparison of policy functions for the Blinder case, using different values of parameter  $\kappa$ .

## 7.2 Approximation and numerical integration methods

The results presented in section 6 as the optimal monetary policy function under the constraint  $E[d(\varepsilon)] = 0$  were computed using a linear approximation of the policy function, while all numerical integration was done using the Simpson numerical integration method. The grid points were  $\varepsilon \in (-14, 14)$  with step of 0.5. This means that the solver finds the minimum of  $E[V(\varepsilon)]$  over 29 points.

As a robustness check, I also used Gaussian quadrature for integration and Chebyshev polynomials as policy function approximation. The number of points in the  $\varepsilon$  grid is more than twice the number of nodes in the Gauss-Hermite quadrature (of 12th order). Using 12th order guarantees the precise value of the integral if the function being integrated can be written as a Hermite polynomial of up to 23rd order. In theory this should be sufficient for any reasonably smooth function. However, it seems that the policy function can have jumps and hence might not be well approximated by polynomials of any order. Therefore the Simpson method seems to be more reliable. However, as the reader can see from figure 11, the differences are usually fairly small.

## 8 Conclusion

The aim of this paper was to model a situation where the public cannot perfectly monitor the central bank behavior. In this setting I showed how this information asymmetry can give rise to a nonlinear reaction function of the central bank characterized by a standard (perfectly symmetric) quadratic loss function. By showing so, this paper contributes to the discussion about nonlinear monetary policy rules and asymmetric preferences of the central bank.

In my model, the central bank announces a policy different from its true preferences in order to decrease the inflationary bias. In this sense, the central bank does not change its preferences, as was implicitly assumed by Rogoff (1985) and Blinder (1998) in their contributions to this literature. The decrease in the inflationary bias is then a result of a reputation building repeated game. Reputation building is nontrivial as the representative agent observes only a noisy signal about the allocation chosen by the central bank. Observing the noisy signal, the representative

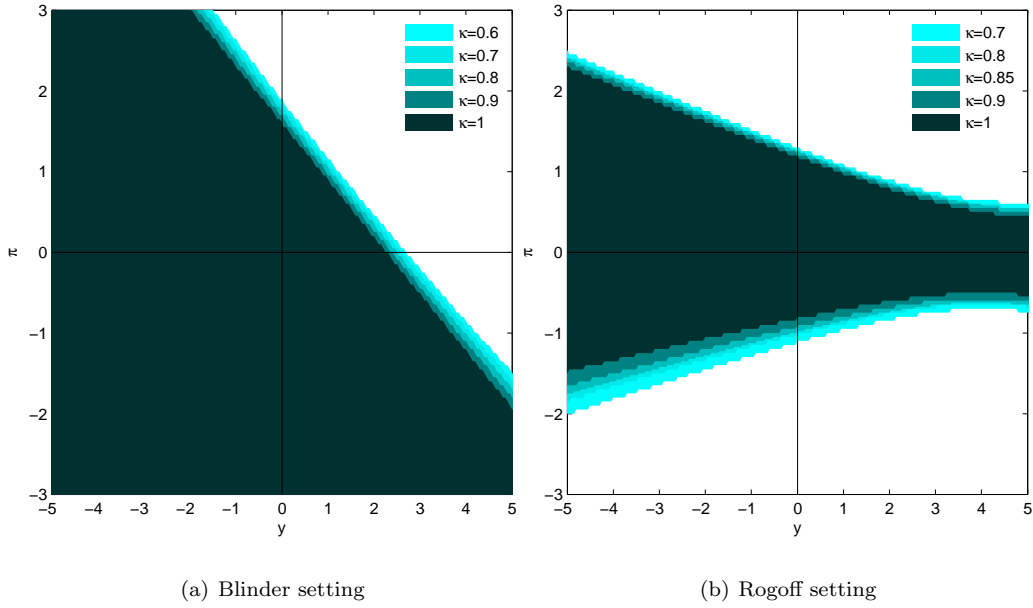


Figure 10: Comparison of the No rejection region  $\mathcal{N}$  for different values of parameter  $\kappa$ .

agent tests the announced policy against the deviation policy and chooses to either believe the central bank's announcement or revise her expectations.

In my model, the central bank exploits the informational advantage which comes from the fact that the representative agent neither perfectly observes neither the allocation chosen by the central bank nor the value of the supply shock.

I analyzed the two famous answers to the inflationary bias from the literature, Rogoff's conservative central bank and Blinder's targeting the natural level of output. In my setting, the optimal monetary policy is nonlinear in both cases. The nature of the nonlinearity is different in the two settings; the central bank is relatively more accommodative for the positive shocks in the Blinder case, and for the negative shocks in the Rogoff case. While the deviation from linearity is rather

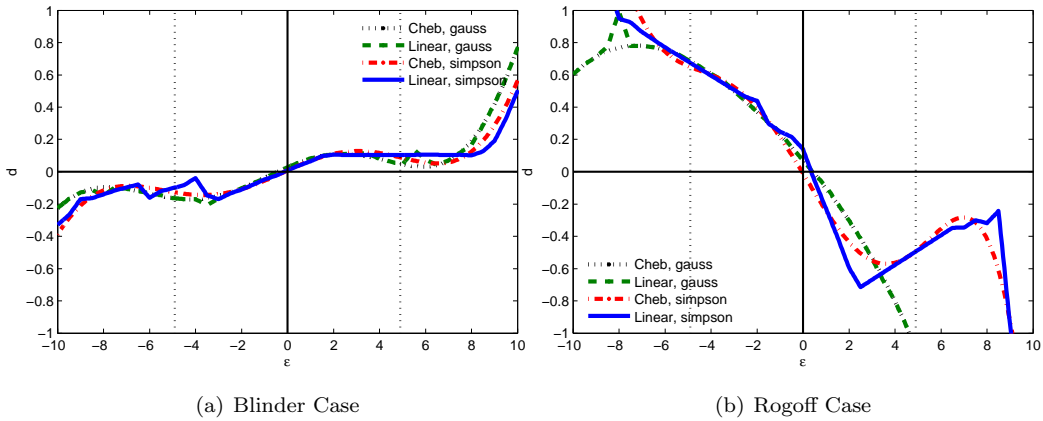


Figure 11: Comparison of different approximation and numerical integration methods

limited in the Blinder case, it is very pronounced in the Rogoff case.

Furthermore, starting from the Blinder case, where the central bank targets the natural rate of output and uses the same output-inflation trade-off parameter  $\beta$  as the rest of the society, the central bank can deliver lower expected social loss by decreasing the output target below zero or by decreasing the trade-off parameter  $\beta$ .

Comparing the two settings, the social welfare is higher in Blinder setting. This means that Rogoff suggestion of how to deal with the problem of inflation bias is less favorable the one of Blinder. The natural next step in this research would be to perform a full grid search to characterize the optimal policy.

Finally, the presented model suggests that it is worthwhile to distinguish between the true preferences and the preferences implied by observed outcomes. My model assumes public knowledge of the true preferences and tries to say something about the “announced” preferences given the economic outcomes. However, in reality the true preferences are not known. For example, different parameterizations of the new keynesian model would give different values of  $y_h^*$  while some other models might give completely different numbers. My model could be then inverted and used to provide insights about the true preferences given the observed outcomes.

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# A Overview of Estimates of Central banks' Preferences

paper	estimated equation	estimate (SE)	method	period
Söderström et al. (2005)	$\min_t \text{var}(\pi) + \lambda \text{var}(x_t) + \nu \text{var}(\Delta i_t)$	$\lambda = 0$ (0)	matching sim. moments	1984Q4-1999Q4
Dennis (2006)	$E_t \sum_{j=0}^{\infty} \beta^j [(\pi_{t+j}^a - \pi^*)^2 + \lambda x_{t+j}^2 + \nu (\Delta i_{t+j})^2]$	$\lambda = 2.94$ (5.685)	Quasi-FIML	1982Q1-2000Q2
		$\lambda_{\pi} = 0.39$ (0.016)		
Ozlalet (2003)	$(1 - \beta) E_t \sum_{j=0}^{\infty} \beta^j [\lambda_{\pi} (\bar{\pi}_{t+j})^2 + \lambda_x x_{t+j}^2 + \lambda_i (\Delta i_{t+j})^2]$ such that $\lambda_{\pi} + \lambda_x + \lambda_i = 1$	$\lambda_x = 0.26$ (0.012)	ML	1970Q1-1999Q1
		$\lambda_i = 0.35$ (0.017)		
Favero and Rovelli (2003)	$E_t \sum_{j=0}^{\infty} \beta^j [(\pi_{t+j} - \pi^*)^2 + \lambda x_{t+j}^2 + \nu (\Delta i_{t+j})^2]$	$\lambda = 0.00125$ (0.0002)	GMM	1980Q3-1998Q3
		$\lambda_c = 0.135$ (0.058)		
Givens (2009)	$E_t (1 - \beta) \sum_{j=0}^{\infty} \beta^j [\pi_{t+j}^2 + \lambda x_{t+j}^2 + \nu (\Delta i_{t+j})^2]$	$\lambda_d = 0.099$ (0.043)	ML	1982Q1-2008Q4

Table 4: Central bank preference parameter estimates

## B Derivations

### Inflation expectations

$$\begin{aligned}\pi_{t|t-1} &= \mathbb{E}[\pi_t^d] = \mathbb{E}\left[\frac{\alpha^2}{\alpha^2 + \beta}\left(\pi_{t|t-1} + \frac{1}{\alpha}y^*\right) - \frac{\alpha}{\alpha^2 + \beta}\varepsilon_t\right] \\ \pi_{t|t-1} &= \frac{\alpha^2}{\alpha^2 + \beta}\left(\pi_{t|t-1} + \frac{1}{\alpha}y^*\right) \\ \pi_{t|t-1} &= \frac{\alpha}{\beta}y^*\end{aligned}$$

**Inflation under discretion** Combining (9) and (6),

$$\begin{aligned}\pi_t^{ree} &= \frac{\alpha^2}{\alpha^2 + \beta}\left(\frac{\alpha}{\beta}y^* + \frac{1}{\alpha}y^*\right) - \frac{\alpha}{\alpha^2 + \beta}\varepsilon_t, \\ \pi_t^{ree} &= \frac{\alpha^2}{\alpha^2 + \beta}\left(\frac{\alpha^2 + \beta}{\alpha\beta}y^*\right) - \frac{\alpha}{\alpha^2 + \beta}\varepsilon_t \\ \pi_t^{ree} &= \frac{\alpha}{\beta}y^* - \frac{\alpha}{\alpha^2 + \beta}\varepsilon_t\end{aligned}$$

### Loglikelihood

$$\begin{aligned}L_{ND}(\varepsilon_t) &\equiv \frac{1}{2}\left(y_t^{ND}(\beta_h, y_l^*) - y_h^*\right)^2 + \frac{\beta_l}{2}\left(\pi_t^{ND}(\beta_h, y_l^*)\right)^2 \\ &= \frac{1}{2}\left(\frac{\beta_h}{\alpha^2 + \beta_h}\varepsilon_t - y_h^*\right)^2 + \frac{\beta_l}{2}\left(\frac{\alpha}{\beta_h}y_l^* - \frac{\alpha}{\alpha^2 + \beta_h}\varepsilon_t\right)^2 \\ &= \frac{1}{2}\left(\frac{\beta_h}{\alpha^2 + \beta_h}\right)^2\varepsilon_t^2 - \frac{\beta_h}{\alpha^2 + \beta_h}y_h^*\varepsilon_t + \frac{1}{2}(y_h^*)^2 \\ &\quad + \frac{\beta_l}{2}\left(\frac{\alpha}{\beta_h}\right)^2(y_l^*)^2 - \beta_l\frac{\alpha^2}{\beta_h(\alpha^2 + \beta_h)}y_l^*\varepsilon_t + \frac{\beta_l}{2}\left(\frac{\alpha}{\alpha^2 + \beta_h}\right)^2\varepsilon_t^2 \\ &= \frac{1}{2}\varepsilon_t^2\left(\left(\frac{\beta_h}{\alpha^2 + \beta_h}\right)^2 + \beta_l\left(\frac{\alpha}{\alpha^2 + \beta_h}\right)^2\right) \\ &\quad - \varepsilon_t\left(\frac{\beta_h}{\alpha^2 + \beta_h}y_h^* + \beta_l\frac{\alpha^2}{\beta_h(\alpha^2 + \beta_h)}y_l^*\right) \\ &\quad + \frac{1}{2}\left((y_h^*)^2 + \beta_l\left(\frac{\alpha}{\beta_h}\right)^2(y_l^*)^2\right)\end{aligned}$$

and in the punished state:

$$\begin{aligned}L^P(\varepsilon_t) &\equiv \frac{1}{2}\left(y_t^D(\beta_h, y_l^*) - y_h^*\right)^2 + \frac{\beta_l}{2}\left(\pi_t^D(\beta_h, y_l^*)\right)^2 \\ &= \frac{1}{2}\left(\frac{\beta_l}{\alpha^2 + \beta_l}\varepsilon_t - y_h^*\right)^2 + \frac{\beta_l}{2}\left(\frac{\alpha}{\beta_l}y_h^* - \frac{\alpha}{\alpha^2 + \beta_l}\varepsilon_t\right)^2 \\ &= \frac{1}{2}\left(\left(\frac{\beta_l}{\alpha^2 + \beta_l}\right)^2\varepsilon_t^2 - 2\frac{\beta_l}{\alpha^2 + \beta_l}y_h^*\varepsilon_t + (y_h^*)^2\right) + \frac{\beta_l}{2}\left(\left(\frac{\alpha}{\beta_l}\right)^2(y_h^*)^2 - 2\frac{\alpha}{\beta_l}\frac{\alpha}{\alpha^2 + \beta_l}y_h^*\varepsilon_t + \left(\frac{\alpha}{\alpha^2 + \beta_l}\right)^2\varepsilon_t^2\right) \\ &= \frac{1}{2}\varepsilon_t^2\left(\left(\frac{\beta_l}{\alpha^2 + \beta_l}\right)^2 + \beta_l\left(\frac{\alpha}{\alpha^2 + \beta_l}\right)^2\right) - \varepsilon_t\left(\frac{\beta_l}{\alpha^2 + \beta_l}y_h^* + \beta_l\frac{\alpha}{\beta_l}\frac{\alpha}{\alpha^2 + \beta_l}y_h^*\right) + \frac{1}{2}(y_h^*)^2\left(1 + \beta_l\left(\frac{\alpha}{\beta_l}\right)^2\right)\end{aligned}$$

allocation when deviation Inflation:

$$\begin{aligned}
\pi_t^D &= \frac{\alpha}{\alpha^2 + \beta_l} y_h^* + \frac{\alpha^2}{\alpha^2 + \beta_l} \pi_{t|t-1} - \frac{\alpha}{\alpha^2 + \beta_l} \varepsilon_t \\
&= \frac{\alpha}{\alpha^2 + \beta_l} y_h^* + \frac{\alpha^2}{\alpha^2 + \beta_l} \frac{\alpha}{\beta_h} y_l^* - \frac{\alpha}{\alpha^2 + \beta_l} \varepsilon_t \\
&= \frac{\alpha}{\alpha^2 + \beta_l} \left( y_h^* + \frac{\alpha^2}{\beta_h} y_l^* \right) - \frac{\alpha}{\alpha^2 + \beta_l} \varepsilon_t \\
\pi_t^D &= \frac{\alpha}{\alpha^2 + \beta_l} \frac{1}{\beta_h} (\beta_h y_h^* + \alpha^2 y_l^* - \beta_h \varepsilon_t)
\end{aligned}$$

Output:

$$\begin{aligned}
y_t^D &= \frac{\alpha^2}{\alpha^2 + \beta_l} y_h^* - \frac{\alpha \beta_l}{\alpha^2 + \beta_l} \pi_{t|t-1} + \frac{\beta_l}{\alpha^2 + \beta_l} \varepsilon_t \\
&= \frac{\alpha^2}{\alpha^2 + \beta_l} y_h^* - \frac{\alpha \beta_l}{\alpha^2 + \beta_l} \frac{\alpha}{\beta_h} y_l^* + \frac{\beta_l}{\alpha^2 + \beta_l} \varepsilon_t \\
&= \frac{1}{\alpha^2 + \beta_l} \left( \alpha^2 y_h^* - \alpha \beta_l \frac{\alpha}{\beta_h} y_l^* + \beta_l \varepsilon_t \right) \\
y_t^D &= \frac{1}{\alpha^2 + \beta_l} \frac{1}{\beta_h} (\alpha^2 \beta_h y_h^* - \alpha^2 \beta_l y_l^* + \beta_l \beta_h \varepsilon_t)
\end{aligned}$$

Expected loss

$$\begin{aligned}
L_t(\beta_l, \beta_h, \varepsilon_t) &= \frac{1}{2} \left( \frac{\alpha}{\beta_h} y^* - \frac{\alpha}{\alpha^2 + \beta_h} \varepsilon_t - y^* \right)^2 + \frac{\beta_l}{2} \left( \frac{\beta_h}{\alpha^2 + \beta_h} \varepsilon_t \right)^2 \\
&= \frac{1}{2} \left[ \left( \frac{\beta_h}{\alpha^2 + \beta_h} \right)^2 \varepsilon_t^2 - 2y^* \frac{\beta_h}{\alpha^2 + \beta_h} \varepsilon_t + (y^*)^2 \right] \\
&\quad + \frac{\beta_l}{2} \left[ \left( \frac{\alpha}{\beta_h} y^* \right)^2 - 2 \frac{\alpha}{\beta_h} y^* \frac{\alpha}{\alpha^2 + \beta_h} \varepsilon_t + \left( \frac{\alpha}{\alpha^2 + \beta_h} \right)^2 \varepsilon_t^2 \right] \\
\mathbb{E}L_t(\beta_l, \beta_h) &= \frac{1}{2} \left[ \left( \frac{\beta_h}{\alpha^2 + \beta_h} \right)^2 \sigma_\varepsilon^2 + (y^*)^2 \right] + \left[ \left( \frac{\alpha}{\beta_h} y^* \right)^2 + \left( \frac{\alpha}{\alpha^2 + \beta_h} \right)^2 \sigma_\varepsilon^2 \right] \\
&= \frac{1}{2} (\beta_h^2 + \alpha^2 \beta_l) \left( \frac{\sigma_\varepsilon^2}{(\alpha^2 + \beta_h)^2} + \frac{(y^*)^2}{\beta_h^2} \right) \\
\frac{\partial \mathbb{E}L_t(\beta_l, \beta_h)}{\partial \beta_h} &= \frac{1}{2} \left( \frac{\sigma_\varepsilon^2}{(\alpha^2 + \beta_h)^2} \right) 2\beta_h + \frac{1}{2} (\beta_h^2 + \alpha^2 \beta_l) \left( -2 \frac{\sigma_\varepsilon^2}{\alpha^2 + \beta_h} - 2 \frac{(y^*)^2}{\beta_h^3} \right) \\
&= \alpha^2 \left( \frac{\sigma_\varepsilon^2 (\beta_h - \beta_l)}{(\alpha^2 + \beta_h)^3} - \frac{(y^*)^2}{\beta_h^3} \beta_l \right)
\end{aligned}$$